Descriptive Geometry

FAUNCE

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DESCRIPTIVE
GEOMETRY.

BY

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DESCRIPTIVE GEOMETRY.

CHAPTER I.

ELEMENTARY PRINCIPLES.

1. DESCRIPTIVE GEOMETRY is the art of representing a definite body in space upon two planes, at right angles with each other, by lines falling perpendicularly to the planes from all the points of the intersection of every two contiguous sides of the body, and from all points of its contour, and of solving all graphical problems involving three dimensions.

2. These planes are called coördinate planes, or the planes of projection, one of which is horizontal and the other vertical. P and Q, Fig. 1, represent two such planes, and their line of intersection GL is called the ground line.

3. The perpendicular lines from a point in space to the planes of projection are called the *projecting lines* of that point. $aa_{,}^{v}$, aa^{h}, Fig. 1, are the projecting lines of the point a.

4. Each coördinate plane is supposed to extend indefinitely in both directions from the ground line; hence they will form by their intersection four dihedral angles (see Fig. 1).

The *first angle* is above the horizontal, and in front of the vertical plane.

The *second* is above the horizontal, and behind the vertical.

The *third* is below the horizontal, and behind the vertical.

The *fourth* is below the horizontal, and in front of the vertical.

The points a, b, c, and d, Fig. 1, are in the first, second, third, and fourth angles respectively.

5. In order that the two projections of an object may be represented on the same sheet of paper, the upper part of the vertical plane is revolved backward, about the ground line, as an axis, until it coincides with the horizontal plane.

This being done, it is evident that all that portion of the plane of the paper which is in front of or below the ground line represents not only that part of the horizontal plane which is in front of the vertical, but also that part of the vertical plane which is below the horizontal, while the portion above the ground line represents that part of the vertical which is above the horizontal, and also that part of the horizontal which is behind the vertical plane.

6. All points in the first and second angles are vertically projected above the ground line; all points in the third or fourth angles below it.

Points in the first and fourth angles are horizontally projected in front of the ground line; those in the second and third behind it.

Thus the points *a*, *b*, *c*, and *d*, situated in the first, second, third, and fourth angles respectively, as shown in Fig. 1, are represented on the plane of the paper, as shown in Fig. 2.

7. Lines situated in either of the four angles are represented by their projections in the same way as described for the point.

8. The two projections of a point are on one and the same straight line perpendicular to the ground line.

9. Two projections are always necessary to definitely locate a point or line in space.

10. The distance of the vertical projection of a point from the ground line is equal to the distance of the point itself, in space, from the horizontal plane; and the distance of its horizontal projection from the ground line is equal to the distance of the point, in space, from the vertical plane.

11. A point situated upon either of the coördinate planes is its own projection on that plane, and its other projection is in the ground line.

12. A right line is determined by two points; hence, any two lines drawn at pleasure, except parallel to each other and perpendicular to the ground line, will represent the projections of a line in space.

13. If two lines are parallel in space, their projections upon either plane will also be parallel.

14. If a right line is perpendicular to either coördinate plane, its projection on that plane will be a point, and its projection on the other plane will be perpendicular to the ground line.

15. If a line be parallel to either plane, its projection on that plane will be parallel to the line itself, and its projection on the other plane will be parallel to the ground line.

16. If a line is parallel to both planes, or to the ground line, both projections will be parallel to the ground line.

17. When a line is parallel to a plane, its projection on that plane will be equal to the true length of the line.

18. If a point be on a line, its projections will be on the projections of the line.

19. The planes which contain the projecting lines of all points in a straight line, — or, in other words, which contain the straight line and are perpendicular to the planes of projection, — are called the *projecting planes* of that line.

20. If two lines intersect in space, their projections must also intersect, and the right line joining the points in which the projections intersect must be perpendicular to the ground line; for the intersection of two lines must be a point, common to both lines, whose projections must be on the horizontal and vertical projections of each of the lines, hence at their intersections respectively.

21. Since two points determine a straight line, it may be determined by its intersections with the coördinate planes. These intersections will be points, and are called the *traces* of the line, horizontal or vertical, according as they are on the horizontal or vertical plane. A line can have but two traces; it may have only one. The points *m* and *g*, Fig. 3, are the traces of the line *gm*, likewise *e* and *f* of the line *ef*.

22. In the same way, a plane being determined by two lines, it may be determined by its intersections with the coördinate planes. These intersections will be lines, and they are called the traces of the plane. A plane can have one or two traces. The line *ab*, Fig. 3, is the vertical trace, and *cd* the horizontal trace of the plane X.

23. The traces of a line lying in a plane must be in the traces

of that plane; for every line, as *ef* and *gm*, Fig. 3, lying in the plane X, must intersect the coördinate planes somewhere on the lines *ab* and *cd* in which the plane X intersects the coördinate planes.

24. The two traces of the same plane must necessarily meet the ground line in the same point; for, by the definition of traces, that point is at once in the given plane and in each of the coördinate planes; hence it is at the intersection of the three planes, *i.e.*, on the ground line.

If the plane be parallel to the ground line, this point is at an infinite distance; hence the two traces are parallel to each other, and to the ground line.

25. If a plane be parallel to either coördinate plane, it will have but one trace, which will be on the other plane and parallel to the ground line.

26. If a plane be perpendicular to either coördinate plane, its trace on the other plane will be perpendicular to the ground line. If it be perpendicular to both coördinate planes, both of its traces will be in a line perpendicular to the ground line. Such a plane is called a *profile* plane.

27. If a plane contains the ground line, its position is not determined.

28. Lines and planes are supposed to be of indefinite length; hence the projections of lines and the traces of planes may always be produced indefinitely in either direction.

29. If a right line, A, is perpendicular to a plane, its projections will be respectively perpendicular to the traces of the plane.

If a plane is perpendicular to two other planes, it will be perpendicular to their line of intersection, and any line of this plane which intersects this line of intersection will be perpendicular to it. But the horizontal projecting plane of the line, A, is perpendicular to H (Art. 19), and it is also perpendicular to the given plane, since it contains a line perpendicular to the given plane; hence it is perpendicular to their line of intersection, which is the horizontal trace of the given plane. But the horizontal projection of the given line passes through this trace, hence it must be perpendicular to it.

In the same way it can be proved that the vertical projection of the line is perpendicular to the vertical trace of the plane.

The converse of this must also be true, — that is, if a plane is perpendicular to a line, the traces of the plane must be respectively perpendicular to the projections of the line

NOTATION.

30. We will designate a point in space by a small letter, and its projections by the same letter with an *h* or *v* written above: thus a^h represents the horizontal, and a the vertical projection of the point a.

31. A line in space will be designated by a capital letter, and its projections by the same letter with an *h* or *v* written above, the same as with the point.

In speaking of a line in space, we may say the line A, or, since a line is determined by two points, the line *ab*.

32. The horizontal coördinate plane will be designated by the capital letter H, the vertical by the capital letter V, and any other plane in space by any other capital letter.

The traces of a plane will be designated by the same letter as the plane, with H or V prefixed: thus, HP denotes the horizontal, and VP the vertical trace of the plane P.

A plane in space is determined by two parallel or intersecting lines, by a line and a point, or by three points; hence, we may speak of a plane as the plane P, the plane (A, B), the plane (A, *a*), or the plane (*a*, *b*, *c*).

33. Given and required lines, if in the first angle, and visible, are represented by full lines in each projection; if in the second, third, or fourth angles, or invisible in the first angle, by short dashes in each projection. Auxiliary lines, — that is, lines which are used simply as a means of obtaining the solution of a problem, — in any angle, are represented by long and short dashes. Construction lines are represented by short dashes. Represent the true length of a line, when found, by long and short dashes.

The V trace, when above GL, and the H trace, when below GL, of a given or required plane are represented by full lines. The V trace, when below GL, and the H trace, when above GL, of these planes are represented by a series of one long and two short dashes. Traces of auxiliary planes, whether above or below GL, are represented by a series of one long and three short dashes.

Represent an angle, when found, by full lines for a short distance from the vertex, as shown in Fig. 14.

EXAMPLES.

Ex. 1. — Draw the projections of a line in the third angle parallel to V, and oblique to H.

Ex. 2. — Of a line in the second angle parallel to H, and oblique to V.

Ex. 3. — Of any line in the fourth angle.

Ex. 4. — Of any line lying in V, and below H.

Ex. 5. — Of a line lying in H, and behind V.

Ex. 6. — Of a line in the third angle, and perpendicular to V.

Ex. 7. — Of a line in the fourth angle, and perpendicular to H.

Ex. 8. — Of a line lying in a profile plane, and oblique to both V and H.

Ex. 9. — Of a line crossing the first, fourth, and third angles.

Ex. 10. — Of a line crossing the second, first, and fourth angles.

Ex. 11. — Of a line crossing the first and third angles.

Ex. 12. — Of a line crossing the second, third, and fourth angles.

Ex. 13. — Of two intersecting lines lying in the third angle.

Ex. 14. — Of two lines lying in the first angle which do not intersect, and are not parallel.

Ex. 15. — Draw the traces of a plane which is oblique to both V and H.

Ex. 16. — Of a profile plane.

Ex. 17. — Of a plane parallel to and behind V.

Ex. 18. — Of a plane perpendicular to V, and oblique to H.

Ex. 19. — Of a plane perpendicular to H, and oblique to V.

Ex. 20. — Of a plane parallel to GL, oblique to both V and H, and crossing the second, first, and fourth angles.

Ex. 21. — Of a plane parallel to GL, oblique to both V and H, and crossing the first, fourth, and third angles.

CHAPTER II.

Problems Relating to the Point, Line, and Plane.

34. Problem 1. — *To find the traces of a given line.*

Let the line be given by its two projections A^v and A^h, Figs. 4 and 5. We will first find the horizontal trace. This, being in H, will have its vertical projection in the ground line (Art. 11), and it must also be on the vertical projection of the line (Art. 18), hence, when the vertical projection of the line, produced if necessary, meets the ground line, as at a^v, we know that the line in space is passing through H at some point, which may be either in front of or behind V, and since the two projections of a point must be in a line perpendicular to GL (Art. 8), and in the two projections of the line (Art. 18) this point will be where a perpendicular to GL from a^v meets the horizontal projection of the line, as a^h. Hence, a^h is the horizontal trace of the line A.

If we put horizontal in the place of vertical, and vertical in the place of horizontal, in the preceding statement, we show how to find the vertical trace of the line, — that is, produce A^h until it meets GL at b^h; erect a perpendicular at that point, and b^v, the point where it intersects A^v, will be the vertical trace of the line.

35. Problem 2. — *Given the traces of a line to construct its projections.*

Let a^h and b^v, Figs. 4 and 5, be the traces; since these points lie in the coördinate planes, their vertical and horizontal projections respectively will be in GL at a^v and b^h (Art. 11); joining a^v with b^v, and a^h with b^h, we have A^v and A^h, the two projections required.

36. Problem 3. — *To find the true length of a line joining two given points in space.*

When a line is parallel to either coördinate plane its projection on that plane is equal to the true length of the line (Art. 17). Hence, we have only to revolve the line about an axis through either end until it is parallel to one of the planes, and find its projection on that plane. Let A^v and A^h, Fig. 6, be the projections of the line joining the points a and b. Revolve the line about a vertical axis through the point a; every point in the line, except the one in the axis, moves in a horizontal circle, of which the horizontal projection is a circle, and the vertical projection a line parallel to GL. When the horizontal projection becomes parallel to GL, the line is parallel to V (Art. 15); a^v and a^h do not move; b^h has described the arc of a circle whose radius is $a^h b^h$, and is found at b_1^h; b^v moves in a line parallel to GL, and is found perpendicularly above b_1^h at b_1^v. Therefore, $a^v b_1^v$ represents the true length of the line.

The line could have been revolved about an axis perpendicular to V, until it was parallel to H, in the same way.

37. *Second method.* — The true length of a line may also be obtained in another way by revolving the horizontal projecting plane of the line about its horizontal trace, into the horizontal plane, when every line of the projecting plane, hence the line in question, will be shown in its true length on H.

Let the line be given as in Fig. 7. The horizontal projecting lines of each end of the line a and b are perpendicular to the horizontal trace $a^h b^h$ of the horizontal projecting plane, therefore they will be so after revolution. Hence, draw the lines $a_1^h a^h$ and $b_1^h b^h$ perpendicular to $a^h b^h$, and make them equal respectively to the lines $a^v r$ and $b^v s$, which represent the heights of the points in space above H; the line joining these two points a_1^h and b_1^h will be the true length required.

This result could have been obtained by revolving the vertical projecting plane until it coincides with V, when $a^v a_1^v$ is made equal to $a^h r$ and $b^v b_1^v$ equal to $b^h s$, the true length being $a_1^v b_1^v$.

38. If the line passes through the plane into which it is revolved, as in Fig. 8, the point in which it pierces the plane being in the axis, does not move, and the two ends of the line revolve in opposite directions; that is, $a^h a_1^h$ is still made equal to $a^v r$, and $b^h b_1^h$ equal to $b^v s$; but they are laid off in *opposite* directions, and $a_1^h b_1^h$, the true length of the line, is found to pass through c^h, the horizontal trace of the line.

39. PROBLEM 4. — *To pass a plane through two intersecting or parallel lines.*

The traces of a line lying in a plane must be in the traces of the plane (Art. 23), hence we have only to find the traces of each line, and a line drawn through the two horizontal traces will be the horizontal trace of the plane, and one through the vertical traces will be the vertical trace of the plane.

Let A and B, Fig. 9, be the two lines intersecting in e. The traces of A are b^h and a^v, and of B are c^h and d^v (Art. 34). HP is drawn through b^h and c^h, and VP through a^v and d^v. They should intersect the ground line in the same point.

40. If we wished to pass a plane through three points, draw two lines through the points, and proceed as described above (see Fig. 10); a, b, and c are the given points.

41. Only one plane can be passed through two lines, but an infinite number can be passed through one line. Thus, in Fig. 11, the planes P, Q, R, S, etc., each contain the line B.

42. PROBLEM 5. — *Given one projection of a line, or point, lying in a plane, to find the other projection.*

Let A^v, Fig. 12, be the vertical projection of a line lying in the plane P, given by its traces VP and HP. The traces of the line must lie in the traces of the plane (Art. 23); therefore, the vertical trace of the line A must be at a^v where A^v intersects VP, and its horizontal projection is at a^h. The horizontal trace must be in a line perpendicular to GL at the point b^v where A^v intersects it, and on HP, hence at their intersection b^h. A^h drawn through a^h and b^h will be the horizontal projection of the line.

If the horizontal projection of the line had been given, the vertical projection would be found in the same way.

43. If one projection of a point were given, we simply draw *any* line through the point, and find the other projection of the line as just described; the other projection of the point will, of course, be on this projection of the line.

44. *If the line is horizontal, its vertical projection is parallel to the ground line, and its horizontal projection is parallel to the horizontal trace of the plane in which it lies.* — In Fig. 13, A^v is the vertical projection of such a line lying in the plane P, a^v and a^h are found as in Art. 42. Since A^v is parallel to GL, b^v and b^h are at an infinite distance from a^v, hence A^h is parallel to HP.

45. Problem 6. — *Given tne projections of a point, or line, lying in a plane, to find its position when the plane shall have been revolved to coincide with either coördinate plane.*

Let (c^v, c^h), Fig. 12, be the projections of a point in the plane P. If we revolve the plane about its horizontal trace until it coincides with H, the point will move in a plane perpendicular to P, and hence to HP, and will be found somewhere in c^hc, and at the same distance from HP that it was in space, which is equal to the hypothenuse of a right angled triangle, of which c^ve (the height of the point above H) is one side, and c^hd (the perpendicular distance of its horizontal projection from HP) is the other; dc is equal to this hypothenuse, hence the point revolves to c.

In revolving a line it is only necessary to find the revolved position of two points, and the position of the line is determined.

In the same figure we have found the revolved position of one point c of the line A, but the point b being in HP does not move in revolution; hence b^hc is the revolved position of the line A.

The position of the point after the plane has been revolved to coincide with V is found in a similar manner. That is, revolve the plane about its vertical trace until it coincides with V, the point will move in a plane perpendicular to P, and hence to VP, and will be found in a line drawn through c^v perpendicular to VP and at a distance from VP equal to the hypothenuse of a right angled triangle of which the distance of the point in front of V is one side, and the perpendicular distance of its vertical projection from VP is the other.

46. Problem 7. — *To find the true size of the angle made by two intersecting lines.*

If we pass a plane through the two lines, and then revolve the plane into one of the coördinate planes, the angle between the lines will be shown in its true size.

Let A and B, Fig. 14, be the given lines intersecting at a. Pass a plane through these lines by Art. 39. Its horizontal trace is HP. It is not necessary to find VP. Revolve this plane about HP into H. The point of intersection a revolves to $a_{,}$; the points c^h and b^h do not move, hence the two lines will be found at $c^ha_{,}$ and $b^ha_{,}$, and δ will be the angle sought.

The angle could also have been found by finding the vertical trace of the plane of the two lines and revolving into V.

If it had been required to find the projections of the line which would bisect the angle made by the two given lines, it would first have been necessary to find the angle between the given lines as described above, then draw a line bisecting this angle, and counter revolve the plane of the two lines, which also contains the bisecting line, to its original position. The projections of the bisector in its new position will be the projections required.

47. Problem 8. — *To find the true size and shape of any plane surface.*

Let the surface be given as in Fig. 15. (In drawing the projections of this surface the complete projection on one plane only can be assumed, while three points of the other projection may be assumed, and the remaining points in this projection must be found according to Arts. 42 and 43.) Pass a plane through it (to do this, pass a plane through any two of its edges, as *ad* and *dc*). HP is its horizontal trace. Revolve this plane about HP into H. The point a is found at $a_{,}^{h}$, b at $b_{,}^{h}$, c at $c_{,}^{h}$, and d at $d_{,}^{h}$. Hence $a_{,}^{h}b_{,}^{h}c_{,}^{h}d_{,}^{h}$ is the true size and shape of the surface *abcd*.

48. Problem 9. — *To find the shortest distance from a point to a line, and draw the projections of it in its position in space.*

Let n, Fig. 16, be the given point, and A the given line. The shortest distance from n to A will be a perpendicular from n on to A. The relative positions of these two lines will only be shown when the plane containing them has been revolved into one of the coördinate planes. Therefore, pass a plane through the line A and point n (to do this, draw a line through n parallel to A). HP is the horizontal trace (the vertical trace is not needed). Revolve this plane about HP into H. The point n revolves to $n_{,}$, the point d, in the line A, to $d_{,}$, a^{h} remains stationary; hence the line A is found at $A_{,}$. From $n_{,}$ drop a perpendicular on to $A_{,}$, and this $n_{,}c_{,}$, is the length required.

To find the projections of this in its correct position, revolve the plane back to its original position; $c_{,}$ revolves back to c^{h}, and $n^{h}c^{h}$ is the horizontal projection. The vertical projection of c^{h} is found at c^{v}, hence $n^{v}c^{v}$ is the vertical projection sought.

49. Problem 10. — *To find the angle which a given line makes with an oblique plane.*

The angle is that made by the line with its projection on the oblique plane. This projection, together with the line itself, and a perpendicular from any point on the line to the plane, form a right angled triangle in which the angle made by the line with the perpendicular is the complement of the angle sought.

Let A, Fig. 17, be the given line, and P the given plane.

From any point *a* of the line drop a perpendicular B on to the plane (Art. 29). Find the angle which A and B make with each other (Art. 46). δ is this angle, and its complement β is the angle sought.

50. Problem 11. — *To find the line of intersection of two planes.*

Since the line of intersection lies in both planes, its horizontal and vertical traces must lie in the horizontal and vertical traces of both planes respectively, hence at their intersections.

Let P and Q, Fig. 18, be the given planes. VP and VQ intersect at a^v, hence a^v is the vertical trace of the line of intersection. b^h is the horizontal trace, since it is the intersection of HP and HQ. The projections A^v and A^h of the line are found from its traces by Art. 35.

The preceding explanation applies equally well to Fig. 19, where the planes are assumed in a different position.

51. If the horizontal traces of the given planes are parallel, as in Fig. 20, a^v is the vertical trace; but, since HP and HQ will never intersect, there will be no horizontal trace, hence the line must be a horizontal of each plane (Art. 44), and A^v will be drawn through a^v parallel to GL, and A^h parallel to HP and HQ.

52. Let the plane P, Fig. 21, be intersected by the horizontal plane Q. Since the line lies in Q and P, it must be a *horizontal* of the plane P. This line is vertically projected in the vertical trace of the plane Q at A^v, and horizontally projected in A^h parallel to HP.

53. Let the planes P and Q be situated so that the horizontal traces do not intersect within the limits of the paper, as in Fig. 22.

Pass a horizontal auxiliary plane X through the two planes; it will cut a horizontal line out of each plane (Art. 52), B out of P, and C out of Q. Where these two lines intersect, at *n*, will be one point of the line of intersection, *a* will be another; hence A is the line of intersection.

If the planes are so situated that neither the vertical nor the horizontal traces intersect within the limits of the paper, Fig. 23, pass two horizontal auxiliary planes, X and Y, through them. The plane X gives the point n on the line of intersection, as described above, and the plane Y the point m; hence the line A, drawn through these two points, is the line required.

Let the planes be so situated that they cross the ground line at the same point, as in Figs. 24 and 25. We still make use of the same principle of using horizontal auxiliary planes. By this means we obtain the point n. The planes crossing the ground line at the same point o, their line of intersection must pass through the ground line at that point; hence o will be a point on both the vertical and horizontal projections of the line. Therefore, A^v and A^h are the projections of the line of intersection.

54. Let both of the planes be parallel to the ground line, but oblique to both V and H, as in Figs. 26 and 27.

In this case we have to make use of a profile plane (Art. 26). This profile plane cuts out of each oblique plane a line which is projected in the traces of the profile plane. The intersection of these lines will be one point in the line of intersection sought. In order to get this point, it is necessary to revolve the profile plane X about its vertical trace into V. The points a^v and b^v, being in the axis, do not move. Every other point in the plane, as c^h and d^h, move in horizontal circles, and are found, after revolution, at $c_{\prime}$ and $d_{\prime}$. The revolved position of the lines will, therefore, be B and C. They intersect at $e_{\prime}^{v}$. In counter revolution this point is found at (e^v, e^h). A^v and A^h, drawn through e^v and e^h respectively, parallel to the ground line, are the projections of the line of intersection of the planes P and Q. Note that, in Fig. 27, the plane P crosses the second angle; also, that when the profile plane is revolved into V, all that part which is behind V rotates in the same direction as that which is in front; hence if d^h rotates to $d_{\prime}$, c^h must rotate to $c_{\prime}$, and $d_{\prime}$ and $c_{\prime}$ are still on opposite sides of the axis as they were before being revolved. In counter revolution $e_{\prime}^{h}$, which is on the same side of the axis as $c_{\prime}$, will revolve to e^h, and the line of intersection is in the second angle.

55. Let one of the planes pass through the ground line.

Both of its traces will be in the ground line, and it will be necessary, in order to determine the plane, to give the angle it makes with V or H.

Let P be given, as shown in Figs. 28 and 29, and Q passing through the ground line and making angle δ with H. Pass the profile plane X through them. It will cut from these planes lines which are shown in their revolved positions at B and C respectively, intersecting at $e_{\prime}^{v}$. In counter revolution this point goes to (e^v, e^h). Joining e^v and e^h with o, we get A^v and A^h as the two projections of the line of intersection. Note particularly the construction in Fig. 29.

56. Problem 12. — *To find where a line pierces a plane.*

Pass any plane through the line, and find its intersection with the given plane. Since this line and the given line are in the same plane, they will intersect, unless they are parallel, and the point where they intersect must be in the given plane, since it is on a line of that plane; hence it is the point where the line pierces the plane.

Let A, Fig. 30, be the given line, and P the given plane. We will take the plane which projects A upon H as the auxiliary plane. Its traces are VX and HX. *ab* is the line of intersection of the planes X and P. The line A intersects the line *ab* at the point *c*, which is, therefore, the point where A pierces P.

Fig. 31 shows the construction, if the vertical projecting plane were used as the auxiliary plane.

57. In Fig. 32 the general method is shown. Pass any plane X through the line A (Art. 41). Its intersection with P is the line *de*. *c*, the intersection of the lines A and *de*, is the point required.

58. Where the plane is determined by two intersecting lines the principle remains the same, but the details of the construction are different.

Let A, Fig. 33, be the given line, and let the plane be determined by the two lines B and C intersecting in the point *a*. First find the line of intersection of the plane projecting A upon H with the plane (B, C). The line C pierces the plane X at a point whose horizontal projection is c^h (where C^h intersects HX). The vertical projection of this point must be in the vertical projection of the line, and perpendicularly above or below it, hence at c^v. In the same way we find that B pierces X at the point *b*. Therefore $b^v c^v$, $b^h c^h$ are the projections of this line of intersection. A intersects this line *bc* at the point *d*, which is, therefore, the point where it pierces the plane (B, C).

59. PROBLEM 13. — *To find the shortest distance (projections and true length) from a point to a plane.*

The shortest distance must evidently be measured along a line from the point perpendicular to the plane.

Let *a*, Fig. 34, be the given point, and P the given plane. From the point *a* drop the perpendicular B upon P (Art. 29). B pierces P at *d* (Art. 56). $B_{\prime}$, the true length of *ad*, is the distance required.

60. Since the definition of the projection of a point on a plane is where a perpendicular through the point pierces the plane, the point *d* is the projection of the point *a* upon the plane P.

Hence, to find the projection of a line upon an oblique plane, it is only necessary to find where a perpendicular from each end of the line pierces the plane, and join the points so found.

Let A be the given line, Fig. 35, and P the given plane. The projection of *a* upon P is *e*, and of *b* upon P is *o*, hence B is the projection of A upon P.

61. PROBLEM 14. — *To pass a plane through a given point parallel to a given plane.*

Since the required plane is to be parallel to the given plane, their traces must be parallel; therefore, the horizontal projection of a horizontal line of the required plane will be parallel to the horizontal trace of the given plane.

Let *a*, Fig. 36, be the given point, and P the given plane. Draw the horizontal line X of the required plane through *a*. X^v and X^h are its projections, X^h being parallel to HP. b^v is the vertical trace of this line. Hence VQ, drawn through b^v parallel to VP, and HQ parallel to HP, are the traces of the required plane.

62. PROBLEM 15. — *To pass a plane through a given point perpendicular to a given line.*

Let *a*, Fig. 37, be the given point, and A the given line. The traces of the plane must be perpendicular to the projections of the line (Art. 29). Draw the horizontal line X of the required plane through *a*. X^v and X^h are its projections, X^h being perpendicular to A^h. b^v is the vertical trace of this line. Hence VP, drawn through b^v perpendicular to A^v, and HP, perpendicular to A^h, are the traces of the required plane.

63. Problem 16. — *To pass a plane through a given line parallel to another given line.*

Let A and B, Fig. 38, be the given lines. Through any point d of the line A draw a line parallel to B. $B_{,}^{v}$ and $B_{,}^{h}$ parallel respectively to B^v and B^h (Art. 13) will be the projections of such a line. The plane P, which contains these lines, A and $B_{,}$ (Art. 39), will contain A and be parallel to B, since it contains a line which is parallel to B.

64. Problem 17. — *To pass a plane through a given point parallel to two given lines.*

Through the given point draw a line parallel to each of the given lines (Art. 13). The plane which contains these two lines (Art. 39) will be the plane required, since it contains a line parallel to each of the given lines.

65. Problem 18. — *To pass a plane through a given line perpendicular to a given plane.*

Let A, Fig. 39, be the given line, and P the given plane. Through any point e of the line A draw a line perpendicular to P. B^v and B^h will be its projections (Art. 29). The plane Q which contains these lines, A and B (Art. 39), will contain A and be perpendicular to the plane P, since it contains a line which is perpendicular to P.

66. Problem 19. — *To construct the projections of the shortest line which can be drawn terminating in two right lines not in the same plane.*

Let A and B, Fig. 40, be the given lines. The required line must be perpendicular to both given lines. Pass a plane through A parallel to B (Art. 63). VP and HP will be its traces. Find the projection of B upon this plane (Art. 60). It being parallel to the plane, its projection on the plane will be parallel to itself. Hence, it will only be necessary to find the projection of one point, as m, which is at r, and D is the projection of B upon P.

From s, the intersection of D and A, draw a perpendicular E to the plane P. It will intersect B at the point t, and be perpendicular to both A and B. $E_{,}$ will be its true length (Art. 37).

67. The line of greatest declivity of a plane is that line which makes the greatest angle with H.

This is the line of intersection ab (Fig. 41) of the plane P with the plane X, which is perpendicular to both P and H.

Let β be the angle ab makes with H. Then $\tan\beta = \frac{a^v a^h}{a^h b^h}$. But $a^h b^h$, perpendicular to HP, is the shortest line that can be drawn from a^h to HP; hence, β is the greatest angle that can be made by any line of the plane P with H.

68. Problem 20. — *To find the angle a given plane makes with either plane of projection.*

We will first find the angle made with H. This angle is the one made by its line of greatest declivity with H.

Let P, Fig. 42, be the given plane. Revolve the plane X, which cuts from P the line of greatest declivity (Art. 67), about its vertical trace into V. a^v being in the axis does not move; b^h moves in the arc of a circle with a^h as a centre, and is found at b, and β is the required angle.

The plane X could just as well have been revolved about its horizontal trace into H. In that case b^h, Fig. 42, would remain stationary, a^v would revolve to a in a line perpendicular to $a^h b^h$, and δ would be the angle required. Of course, β and δ should be equal.

If the auxiliary plane X were taken perpendicular to V and P, it would cut from P the line making the greatest angle with V, which, being revolved into one of the planes of projection, would show the angle the plane makes with V.

In Fig. 43, β is the angle P makes with V, and δ the angle it makes with H.

69. If one trace of a plane and the angle it makes with that coördinate plane were given, we would determine the other trace by the reverse of the last problem. That is, if HP and δ, Fig. 42, were given, to find VP; draw any line, as $a^h b^h$ perpendicular to HP; draw ab^h making angle δ with $a^h b^h$, and aa^h perpendicular to $a^h b^h$; rotate this triangle about HX as an axis until it is perpen-

dicular to H; *a* is found at a^v, and must be one point of VP; join a^v and the point where HP intersects the ground line, and VP is obtained.

70. Problem 21. — *Given the angle a plane makes with each coördinate plane to construct its traces.*

A plane is tangent to a sphere at a single point. If we pass an auxiliary plane through this point, the centre of the sphere, and perpendicular to H, it will cut a great circle from the sphere and a line from the tangent plane, which will be tangent to the circle. But this line is the line of greatest declivity of the tangent plane, since it is a line cut from it by a plane perpendicular to H and to the tangent plane as it contains a normal to the tangent plane. If this auxiliary plane be revolved about its vertical trace to coincide with V, the line and circle will be shown in their true size and position, and also the true size of the angle the plane makes with H.

An auxiliary plane passed through the tangent point, the centre of the sphere, and perpendicular to V, cuts from the sphere a great circle and from the tangent plane a line tangent to the circle, and which makes the greatest angle with V, and this being revolved into H shows the size of the angle the plane makes with V, etc.

In Fig. 44, with *o* as centre, and any radius describe a circle. This will represent the revolved position of both sections of the sphere. Draw *ab* tangent to this circle so that the angle *abo* is equal to β, the angle the plane makes with H. This line is the revolved position of the line of greatest declivity of the plane, and *a* will be one point on the vertical trace of this plane. Draw *dc* tangent to the circle, making the angle *cdo* equal to δ, the angle which the plane makes with V. This line is the revolved position of the line of the plane which makes the greatest angle with V, and *c* will be one point on the horizontal trace of this plane.

In the counter revolution of *ab*, *a* remains stationary and *b* moves in the arc of a circle of which *o* is the centre until *ab* is perpendicular to the horizontal trace of the plane. Therefore, with *o* as a centre, and *ob* as a radius, describe a circle, and through *c*, tangent to this circle, draw HP, which will be the horizontal trace required. The line VP, joining *a* and the point where HP crosses the ground line, is the vertical trace of the same plane. VP could be constructed independently by drawing it through *a* tangent to the circle drawn with *o* as a centre, and *od* as a radius.

In every case the sum of the angles made by a plane with each plane of projection must be equal to, or greater than, 90 degrees.

71. Problem 22. — *To find the angle made by two oblique planes.*

From any point in space let fall a perpendicular to each plane. The angle made by these two lines will be the angle required.

Another method. — Pass a plane perpendicular to the line of intersection of the two planes; it will be perpendicular to both of these planes, and it will cut a line from each perpendicular to the line of intersection. The angle made by these two lines with each other (Art. 46) will be the angle required.

72. Problem 23. — *To draw the projections of any solid, of definite size, occupying a fixed position in space, resting with its base on an oblique plane which makes known angles with both coördinate planes, and construct the projections of its shadow upon this oblique plane.*

This problem is merely an application of several of the preceding problems.

In Fig. 45, let β be the angle the oblique plane makes with H, and δ the angle it makes with V. Find VP and HP by Art. 70. Let the solid be a rectangular prism, the lowest point of which is at a distance equal to a^vr above H, and a^hr in front of V. The intersections a^v and a^h of the two projections respectively of the horizontal line X of the plane P, which is at a distance above H equal to a^vr, and the line R of the plane P, which is at a distance in front of V equal to a^hr, are the projections of this lowest point. Having found the projections of one point of the base, it is necessary to revolve the plane P into one of the coördinate planes in order to show the base of the prism in its true size and position. The point *a* revolves to $a_{,}$ (Art. 45). Make $a_{,}b_{,}c_{,}d_{,}$ equal to the true size of the base of the prism, and in the desired position. In counter revolution these points will be found in horizontal projection at a^h, b^h, c^h, and d^h respectively. The vertical projections of these points are found at a^v, b^v, c^v, and d^v (Art. 42). This being a right prism, the elements are perpendicular to the plane, hence their projections are perpendicular respectively to HP and VP (Art. 29). To find n^h, the horizontal projection of the top end of the element *an*, revolve the plane projecting this element upon H, with its line of intersection with P, into H. The line of intersection revolves to l^ht, the point *a* to *s*, the element *an* to $sn_{,}$, perpendicular to l^ht, $sn_{,}$ being equal to the real height of the prism. In counter revolution $n_{,}$ goes to n^h. Making the other elements of the same length as a^hn^h, and joining the tops, the horizontal projection of the prism is completed; e^v, m^v, n^v, and o^v are on the vertical projections of the elements vertically above e^h, m^h, n^h, and o^h.

73. Since the definition of the shadow of a point upon a plane is where a ray of light through that point pierces the plane, to find the shadow of this prism upon P, it is only necessary to find where the ray of light, as L, through the point *e* pierces P (Art. 56) at $e_{,}$: and do this for the other points which cast shadows.

CHAPTER III.

PRINCIPLES AND PROBLEMS RELATING TO THE CYLINDER, CONE, AND DOUBLE CURVED SURFACES OF REVOLUTION.

74. A cylinder may be generated by a straight line, called the *generatrix*, moving along a curved line, called the *directrix*, with all its positions parallel. The different positions of the generatrix are called *elements*. If the curve has a centre, a line drawn through it, and parallel to the elements, is called the *axis* of the cylinder.

Any curved section of a cylinder made by a cutting plane, and taken as the limit of the elements, may be called the *base* of the cylinder. If the plane is perpendicular to the axis, the section is a *right section*, and, if this section be taken as the base, the cylinder is called a *right* cylinder. If the right section is a circle, the cylinder is a *cylinder of revolution*.

A cylinder may be called circular or elliptic, according as its right section is a circle or an ellipse.

75. The cone differs from the cylinder only in the fact that its elements, instead of being parallel, pass through a common point called the *vertex*.

The statements and definitions relating to the cylinder, in the preceding article, are equally applicable to the cone.

76. The cylinder and cone must be represented in projection by their lines of apparent contour, whatever other lines may be drawn on their surfaces.

These surfaces are called single curved surfaces.

77. A double curved surface of revolution, such as a sphere, ellipsoid, torus, etc., is generated by a plane curve revolving about a right line contained in its plane. This line is called the *axis* of the surface, and the generating curve is called the *meridian line*, and its plane the *meridian plane*. When a meridian plane is parallel to V, it is called the *principal meridian plane*.

Every point in the meridian line describes the arc of a circle, the centre of which is in the axis of the surface. This circle is called a *parallel*. Hence any plane perpendicular to the axis of a surface of revolution will cut a parallel from the surface.

78. The simplest way of representing a double curved surface of revolution is to assume the axis perpendicular to H, when the greatest parallel will be its horizontal projection, and the principal meridian will be its vertical projection.

79. A plane which contains two lines that are tangent to a surface at a common point will be tangent to the surface, and will, moreover, contain every line which is tangent to the surface at that point.

In the case of a cylinder and cone, the tangent plane must contain an element of the surface.

80. If a single curved surface and its tangent plane be intersected by any secant plane, the line cut from the tangent plane will be tangent to the curve cut from the surface. Hence if this secant plane happens to be the horizontal coördinate plane, the horizontal trace of the tangent plane must be tangent to the base of the surface at the point where the element of contact pierces H.

81. A *normal* to a surface at any point is a right line perpendicular to the tangent plane at that point.

A *normal plane* is a plane which contains the normal line, hence it will be perpendicular to the tangent plane.

82. A plane which contains one line which is tangent to a surface, and is perpendicular to the normal at that point, must be tangent to the surface at that point.

83. PROBLEM 24.—*Given one projection of a point on the surface of a cylinder, to find its other projection; and, second, to pass a plane tangent to the cylinder through this point.*

Let the cylinder be given as in Fig. 46, and let a^h be the horizontal projection of a point on its surface.

Through a^h draw E^h parallel to A^h. This will be the horizontal projection of two elements, one on the upper and one on the lower side of the cylinder. The base of the cylinder, since it rests on H, is the locus of the horizontal traces of all the elements; hence d^h and e^h, where E^h intersects the circle of the base, are the horizontal traces of the two elements, and d^v and e^v are their vertical projections. E^v and $E_{\prime}^{v}$ drawn through d^v and e^v respectively parallel to C^v, are the vertical projections of the two elements horizontally projected in E^h. a^v and $a_{\prime}^{v}$, where a line through a^h perpendicular to GL intersects E^v and $E_{\prime}^{v}$, are the vertical projections of the two points horizontally projected in a^h.

Second. To draw a plane tangent to the cylinder at the point (a^v, a^h). — HP, drawn tangent to the base of the cylinder at d^h, will be the horizontal trace of the tangent plane (Art. 80). The line X will be a horizontal of the plane P; its vertical trace is m^v; VP, drawn through m^v and the point where HP intersects the ground line, is the vertical trace of the tangent plane. The vertical trace of the plane must, of course, pass through the vertical trace of the element of contact, and the vertical trace of the tangent plane could have been found in that way. In finding the plane Q tangent to the cylinder at the point $(a_{\prime}^{v}, a^h)$, since HQ does not intersect GL within the limits of the paper, it was necessary to use both methods to find VQ.

84. Fig. 47 shows the construction when the axis of the cylinder is parallel to both V and H.

Let a^h be the horizontal projection of a point on its surface. It is here necessary to make use of a profile plane. To avoid confusing the figure, assume the profile plane X, containing one end of the cylinder, and revolve it away from the figure about either trace into one of the coördinate planes; in this case about VX into V. It cuts out of the cylinder a right section — a circle in this case — of which c is the centre; c rotates to $c_{\prime\prime}^{v}$, and the circular section with it. The horizontal projection of the element E^h through the point a meets the profile plane at a point whose horizontal projection is b^h. In revolution b^h moves to $b_{\prime\prime}^{h}$; erecting a perpendicular at $b_{\prime\prime}^{h}$, it cuts the circle at $b_{\prime\prime}^{v}$ and $b_{\prime\prime\prime}^{v}$. These are the revolved positions of the points in which the two possible elements through a^h meet the plane X. In counter revolution they are found at b^v and $b_{\prime}^{v}$ respectively. Through these points draw the elements E^v and $E_{\prime}^{v}$. These will be the vertical projections of two elements horizontally projected at E^h; a^v and $a_{\prime}^{v}$, where a line through a^h, perpendicular to GL, intersects E^v and $E_{\prime}^{v}$, are the vertical projections of the two points horizontally projected at a^h.

To draw a plane tangent to the cylinder at the point (a^v, a^h).— The profile plane cuts out of the plane, tangent to the cylinder along the element (E^v, E^h), a line which is tangent to the curve cut from the cylinder at the point (b^v, b^h) where the element E pierces X (Art. 80). Hence T, tangent to the circle at $b_{\prime\prime}^v$, is the revolved position of this line of intersection. In counter revolution d^v does not move, and $e_{\prime}^h$ revolves to e^h; d^v and e^h are the traces of this line which lies in the tangent plane; and, since the tangent plane contains the element E which is parallel to both V and H, VP and HP, drawn through d^v and e^h respectively, parallel to GL, are the traces of the tangent plane.

In the same manner the tangent plane through the point ($a_{\prime}^v$, a^h) may be drawn. Note here that, in counter revolution, $m_{\prime}^h$ revolves to m^h, and VQ and HQ are both found above the ground line, which indicates that the plane crosses from the first, through the second, and into the third angle.

85. PROBLEM 25.— *To pass a plane tangent to a cylinder, through a given point without the surface.*

Since the plane is to be tangent to the cylinder it must contain an element of the cylinder, and its horizontal trace must be tangent to the base (Art. 80). Therefore through the given point draw a line parallel to an element of the cylinder; through the horizontal trace of this auxiliary line draw a line tangent to the base of cylinder. This will be the horizontal trace of the required tangent plane. Its vertical trace is found by means of the vertical trace of the auxiliary line or element of contact.

86. PROBLEM 26.— *To pass a plane tangent to a cylinder, parallel to a given line.*

Through any point of the given line draw an auxiliary line parallel to an element of the cylinder. The plane of the given and auxiliary lines will be parallel to the required plane.

87. PROBLEM 27.— *Given one projection of a point on the surface of a cone to find its other projection; and, second, to pass a plane tangent to the cone through this point.*

Let the cone be given as in Fig. 48, and let a^v be the vertical projection of a point on its surface. B^v will be the vertical projection of two elements passing through a; B^h and $B_{\prime}^{h}$ will be the horizontal projections of these elements, and a^h and $a_{\prime}^{h}$ will be the horizontal projections of the two points vertically projected at a^v.

Second. To draw a plane tangent to the cone at the point (a^v, a^h). — Here, as in the case of the cylinder, HP, drawn tangent to the base of the cone at c^h, will be the horizontal trace of the tangent plane (Art. 80), and the trace of the horizontal Y determines one point s^v in VP. VP could have been determined by finding the vertical trace d^v of the element of contact B.

VQ and HQ, found in the same way, are the traces of a plane tangent to the cone at the point $(a\,, a_{\prime}^{h})$.

88. Fig. 49 shows the construction when the axis of the cone is parallel to both V and H. Let n^v be the vertical projection of a point on its surface. Pass the profile plane X through the base of the cone, and revolve it about VX into V. G^v is the vertical projection of the elements through n. In revolution d^v moves to $d_{\prime}^{v}$ and $d_{\prime\prime}^{v}$, and in counter revolution they are found in plan at d^h and $d_{\prime}^{h}$; G^h and $G_{\prime}^{h}$ drawn through d^h and $d_{\prime}^{h}$ and the vertex o^h, are the horizontal projections of the two elements vertically projected in G^v, and n^v and $n_{\prime}^{h}$ are the horizontal projections required.

In finding the traces of the tangent plane through the point (n^v, n^h), we find one point in each trace, — that is, c^v and m^h, just the same as in Fig. 47; another point in each trace is obtained, of course, by finding the two traces of the element of contact G; joining the horizontal and vertical traces, respectively, we have VP and HP. The plane Q is found in the same way.

If the cone had been so situated that the traces of the element of contact did not fall within easy reach, we would have made use of another profile plane, getting two more points, as we did c^v and m^h.

89. PROBLEM 28. — *To pass a plane tangent to a cone, through a given point without the surface.*

Since the plane is tangent to the cone it must contain its vertex. Therefore, through the vertex and given point draw an auxiliary line; find its horizontal trace and proceed as in Prob. 25.

90. PROBLEM 29. — *To pass a plane tangent to a cone, parallel to a given line.*

Through the vertex of the cone draw an auxiliary line parallel to the given line, and proceed as in Problems 25 and 28.

91. PROBLEM 30. — *Given one projection of a point on the surface of a double curved surface of revolution to find its other projection, and to pass a plane tangent to the surface through this point.*

Let the surface be a torus, represented as shown in Fig. 50 (Arts. 77 and 78), and let n^h be the horizontal projection of a point on its surface. Pass the meridian plane Y through n; HY, drawn through n^h and A^h, is its horizontal trace; revolve this plane into the position of the principal meridian plane; n^h revolves in the arc of a circle to s^h, which, being in the principal meridian, is vertically projected at s^v and $s_{\prime}^v$; in counter revolution s^v and $s_{\prime}^v$ move to n^v and $n_{\prime}^v$ vertically above n^h. Hence n^v and $n_{\prime}^v$ are the vertical projections of two points on the surface of the torus horizontally projected at n^h.

Second. To draw a plane tangent to the torus at the point n. — This plane must contain a line tangent to the surface, and be perpendicular to a normal at that point (Art. 82). $T_{\prime}^v$, drawn tangent to the principal meridian at s^v, will be the revolved position of the tangent line, and $N_{\prime}^v$, drawn through s^v and the centre of the generating circle, the revolved position of the normal. In counter revolution the points m and o do not move, being in the axis; $N_{\prime}^v$ moves to N^v, and $T_{\prime}^v$ to T^v. Both the tangent and normal, since they are in the meridian plane, are horizontally projected in HY. g^h is the horizontal trace of T. Since the tangent plane is to contain T and be perpendicular to N, HP, drawn through g^h perpendicular to N^h, and VP, drawn through the point where HP intersects GL, and perpendicular to N^v, are the traces of the tangent plane.

A point r^v in VP could have been found by means of the horizontal line X of the plane P, and still another by finding the vertical trace f^v of the tangent line.

92. PROBLEM 31. — *To pass a plane through a given line and tangent to a sphere.*

If we pass an auxiliary plane through the centre of the sphere and perpendicular to the given line, it will cut a great circle from the sphere and a line, tangent to this circle, from the required tangent plane. This line will intersect the given line at the point where the given line intersects the auxiliary plane. The plane which contains this tangent line and the given line is the required plane.

The traces of the auxiliary plane are found by Art. 62; the point where the given line pierces this plane is found by Art. 56; then revolve the auxiliary plane, together with its intersection with the given line, and the centre of the sphere, to coincide with either V or H by Art. 45; through the revolved position of the point of intersection draw a line tangent to the circle, which is drawn with the revolved position of the centre of the sphere as a centre, and a radius equal to the radius of the sphere; counter revolve the plane, together with the tangent line just found, to its original position; then pass a plane through the tangent line and the given line, which will be the required plane.

CHAPTER IV.

INTERSECTION OF PLANES AND SOLIDS, AND THE DEVELOPMENT OF SOLIDS.

93. To find the intersection of any solid with any secant plane, pass a series of auxiliary planes through the solid and the secant plane. They will cut lines (straight or curved) from the solid, and straight lines from the secant plane, the intersections of which are points of the required curve of intersection.

This is a general statement applicable alike to prisms, pyramids, cylinders, cones, or double curved surfaces of revolution. While the auxiliary planes may be taken in any position, yet, for simplicity, they should be chosen in such a position as to cut the simplest curve from the solid, — that is, straight lines or circles if possible.

In case of any solid having rectilinear elements, as the cylinder, cone, prism, pyramid, etc., the above process is practically the same as finding where a certain number of elements, or edges, pierce the secant plane (Arts. 56, 57, and 58), since auxiliary planes may be chosen so as to cut elements, or edges, from the solid.

94. The true size of the section can always be found by revolving it about the trace of the secant plane into, or parallel to, one of the coördinate planes (Arts. 45, 46, and 47).

95. The tangent line to the curve of intersection at any point is the intersection of the plane tangent to the surface at that point with the secant plane; for the line, to be tangent to the curve, must lie in the plane of the curve, — that is, the secant plane, and also in the plane tangent to the surface at that point; hence at their intersection.

96. To develop a body is to find, on a flat surface, the true size and shape of the surface, or covering, of the object. Single

curved surfaces and solids bounded by planes can be developed; double curved surfaces cannot be, except approximately.

Solids bounded by planes are developed by finding the true size and shape of each successive face.

Single curved surfaces, as the cylinder, cone, etc., are developed by placing one element in contact with a plane, and rolling the solid until every element has touched this plane. That portion of the plane covered by the solid in its revolution is the development of the solid. When a single curved surface is developed, any curved line upon it will be developed into some line, whose length will be equal to that of the original curve.

CYLINDERS.

97. To find the intersection of any cylinder with an oblique plane, assume the auxiliary planes so that they will cut elements from the cylinder, and the general statement for the intersection of any solid with a plane (Art. 93) becomes for the cylinder as follows: Pass a series of auxiliary planes through the cylinder parallel to its axis and perpendicular to one of the coördinate planes. Each auxiliary plane cuts two elements from the cylinder (except the two tangent planes, which cut but one each) and a right line from the secant plane. The intersections of this line and elements give two points of required curve.

98. Problem 32. — *To find the intersection of a right cylinder with a circular base with an oblique plane, to draw a tangent to this curve at any point, to find the true size of the section, to develop the cylinder, to trace upon the development the curve of intersection, and to draw a tangent to the developed curve.*

Let the cylinder be located as shown in Fig. 51, and P be the secant plane. The auxiliary plane X is parallel to the axis and perpendicular to H (Art. 97); it cuts the two elements C and D from the cylinder and the line X from P; X intersects C at *a*, and D at *b*, which are two points of the curve. Other points could be determined in the same way.

The rest of the points were found by assuming the auxiliary planes as horizontal, which may be a trifle shorter, for this case, than the first method. The horizontal auxiliary plane R cuts out of the cylinder a horizontal circle, and out of the plane the horizontal line R; this line intersects the circle at the points c and d, which are, therefore, two points of the curve. In the same way the plane S gives the points g and k, and the plane Z the points m and n, etc.

To draw a tangent to the point n in the curve.—The tangent line to the curve at the point n is the intersection of the plane tangent to the cylinder at that point and the plane P (Art. 95). VT and HT are the traces of the plane tangent to the cylinder at the point n (Art. 83). T^v and T^h, the intersection of the planes P and T, are the projections of the tangent line. Since the point n must be one point in the tangent line, it is only necessary to find one other point, as r, and the tangent line is determined,—that is, it is not necessary to find the vertical trace of the tangent plane, which shortens the construction.

To find the true size of the section.—Each point in the curve, as a, c, g, m, b, etc., revolve to $a_{\prime}$, $c_{\prime}$, $g_{\prime}$, $m_{\prime}$, $b_{\prime}$, etc. (Art. 45), and the true size of the section is shown.

The tangent line T revolves with the section. The point r^h does not move, since it lies in the axis HP, and the point n revolves to $n_{\prime}$; hence T revolves to $T_{\prime}$.

To develop the cylinder.—If we place one element of the cylinder, as D, Fig. 52, on a plane, and roll the cylinder until the same element coincides with the plane again, that part of the plane covered in this revolution will be the development (Art. 96). The element will, of course, be seen in its true length, and a right section, being perpendicular to the elements, unrolls in a straight line perpendicular to the elements, and equal in length to the circumference of the right section. Hence, $b_{\prime}^{h}n_{\prime}^{h}$, $n_{\prime}^{h}k_{\prime}^{h}$, $k_{\prime}^{h}d_{\prime}^{h}$, etc., Fig. 52, equal to b^hn^h, n^hk^h, k^hd^h, etc., Fig. 51, laid off on a straight line will be the development of the base, and perpendiculars erected at these points will be the position of the several elements; $b_{\prime}^{h}s$ is made equal to the height of the cylinder, and st, drawn parallel to $b_{\prime}^{h}b_{\prime}^{h}$, completes the development of the whole cylinder.

To trace upon the development the curve of intersection.—Make $b_{\prime}^{h}b_{\prime}^{v}$, $n_{\prime}^{h}n_{\prime}^{v}$, $k_{\prime}^{h}k_{\prime}^{v}$, etc., equal to the heights of the points b, n, k, etc., of the curve above H. The curve traced through these successive points will be the development of the curve of intersection.

To draw the tangent line to the point n on the development. — The point n_1^v will be one point of the tangent line sought. The point r being in the plane of the base must be found in the line on which the base unrolls, and at a distance from n_1^h equal to the distance of the point r from the foot of the element along which the plane is tangent, — that is, make $n_1^h r_1^h$ equal to $n^h r^h$, and joining r_1^h and n_1^v we have the developed position of the tangent line. This line must be tangent to the developed position of the curve.

99. PROBLEM 33. — *To find the intersection of an oblique cylinder with any oblique plane, to draw a tangent to any point of the curve, to find the true size of the section, to develop that part of the cylinder between the base and the secant plane, and to draw a tangent to the developed curve.*

Let the cylinder be given as in Fig. 53, and let P be the cutting plane.

Assume the auxiliary planes, R, S, U, etc., parallel to the axis and perpendicular to H (Art. 97). The plane U cuts two elements, F and N, from the cylinder, and the line U from the plane; x and m, the intersections of this line with these elements, are two points of the curve. Similarly the plane R gives the point o; S the points n and p; W gives l and r; X gives k and s; Y gives g and t; and Z the point f. In the construction of the different lines of intersection R, S, U, etc., it is only necessary to construct one of the lines; the others are drawn through the vertical projections of the horizontal traces, parallel to the first, since the lines of intersection of several parallel planes with an oblique plane must be parallel to each other.

The tangent line is found as explained in the last problem. T is the plane tangent to the cylinder along the element through t, and T^v and T^h, the line of intersection of the planes T and P, are the projections of the tangent line to the curve at the point t. The true size of the section is found as explained in the last problem.

To develop that part of the cylinder between the base and the secant plane. — Since neither the plane of the base nor the plane of the section just found is perpendicular to the axis, the sections are not right sections, hence they will not unroll in straight lines. Consequently the first step necessary in the solution of this part of the problem is to find a right section. (K^v, K^h), the section cut by the plane Q, which is perpendicular to the axis, is a right section; K' is the true size of this section; and $K_{\prime}$, Fig. 54, is its development, its length being equal to the periphery of the section K'. Find the true lengths of the successive elements between this right section and the base, and lay them off perpendicular to the line $K_{\prime}$ on lines which represent the developed position of the several elements. The curved line $A_{\prime}A_{\prime}$ drawn through the points thus found is the development of the periphery of the base.

To develop the section cut out by the plane P, find the true lengths of the successive elements between this section and the base, laying them off from the curved line $A_{\prime}A_{\prime}$, and the curved line $l_{\prime}m_{\prime}n_{\prime}$, etc., is the development of this section. This curve could also have been found by finding the true lengths of the elements between the right section and the section cut by P, and laying these lengths off from the development $K_{\prime}$ of the right section. Hence the surface included between the curved lines $A_{\prime}A_{\prime}$ and $l_{\prime}m_{\prime}n_{\prime}$, etc., is the development of that portion of the cylinder between the plane P and the base.

To draw the tangent line to the point t on the development. — The point e is at a certain known distance from the point t, and also a certain other known distance from the point w; hence with $w_{\prime}$, Fig. 54, as a centre and a radius equal to $w^h e^h$, Fig. 53, describe an arc; with $t_{\prime}$, Fig. 54, as a centre and a radius equal to the *true length* of the line te, Fig. 53, describe another arc; the point $e_{\prime}$, where these two arcs intersect, will be one point of the developed position of the line. $t_{\prime}$ must be another, therefore $T_{\prime}$, drawn through $t_{\prime}$ and $e_{\prime}$, is the tangent sought.

CONES.

100. The general statement for the intersection of any cone with any plane is as follows: Pass a series of auxiliary planes through the vertex of the cone, and perpendicular to one of the coördinate planes. These planes cut elements from the cone and lines from the secant plane, the intersections of which give points on the curve.

101. PROBLEM 34. — *To find the intersection of a right cone with a circular base with an oblique plane, etc., as in Problem* 32.

Let the cone and cutting plane P be given as in Fig. 55. The auxiliary plane R passes through the vertex and is perpendicular to H (Art. 100); it cuts the two elements A and B from the cone, and the line R from P; R intersects A at c, and B at g, which are two points of the curve. These two points are the highest and lowest points of the curve, since the line R is the line of greatest declivity of the plane P. Other points could be determined in the same way.

The rest of the points were found by assuming horizontal auxiliary planes X, Y, and Z. The plane X cuts out of the cone a horizontal circle, and out of the plane the horizontal line X; this line intersects the circle at the points d and p, which are, therefore, two points of the curve. In the same way the plane Y gives the points e and n, and the plane Z the points f and m.

The tangent line T, and the true size of the section, are found in the same way as described for the cylinder in Problem 32.

To develop the cone and trace on the development the curve of intersection. — In a right cone all the elements pass through the vertex, and are of the same length, hence the vertex remains stationary and the base unrolls in the arc of a circle, of which the vertex is the centre, and the slant height of the cone (that is, the true length of the element) is the radius. The length of this arc must be equal to the circumference of the base. Therefore, with o', Fig. 56, as a centre, and a radius equal to the true length of an element, describe an arc D' equal in length to the circumference of the base, and $o't'D't'$ is the development of the surface of the cone. Particular elements may be drawn on the development by laying off distances on the arc D' equal to the distance apart of the elements at the base of the cone, and joining these points with the vertex o'. To find the development of the curve of intersection, lay off from o', Fig. 56, on its corresponding element the true length of the element from the vertex to the section, and joining the points thus found, g', f', e', etc., gives the development sought.

To draw the tangent to the development at the point p. — The line C, tangent to the base of the cone, is perpendicular to the element so, hence it will be so after development. Therefore draw C', Fig. 56, perpendicular to $s'o'$, and lay off on it $s'r'$ equal to s^hr^h, Fig. 55; r is also a point of the tangent line T, hence r' must be one point of its developed position, and p' must be another point, consequently T' drawn through r' and p' is the tangent required.

102. PROBLEM 35. — *To find the intersection of an oblique cone with any oblique plane, etc., as in Problem* 33.

Let the cone be given as in Fig. 59, and let P be the cutting plane.

Assume the auxiliary planes R, S, X, etc., through the vertex and perpendicular to H (Art. 100). The plane S cuts two elements, A and B, from the cone, and the line S from the plane; g and r, the intersections of this line with these elements, are two points of the curve. Similarly the plane R gives the point f; X the points k and p; Y the points l and n; and Z the point m.

In the construction of the different lines of intersection R, S, X, etc., it is not necessary to go through the whole process of finding each of them. In the case of the cylinder, these lines of intersection were parallel; here they intersect in a common point o_l. The auxiliary planes, being all perpendicular to H, and passing through the vertex of the cone, will intersect each other in a common line, which must be perpendicular to H, and contain the vertex; D^v will be the vertical projection of this line, and o^h its horizontal projection; hence the point o_l must be on this line, which is common to all the auxiliary planes, and on the plane P, therefore at the point where they intersect. o^h is the horizontal projection of this point, and o_l^v, constructed as explained in Problem 5, is its vertical projection. This point could be found by extending one line of intersection, found regularly, until it intersects the vertical line joining the projections of the vertex.

The tangent line T at the point l, the true size of the section, etc., are found as explained in a previous problem.

To develop that part of the cone which is between the base and the secant plane. — Since the vertex remains stationary, we have a means of locating the developed positions of the elements with respect to each other without getting a right section as in the cylinder.

Draw any right line $o't'$, Fig. 60, and on it lay off the true length of any one of the elements, as E, Fig. 59. Find the true length of the next element A; with this length as a radius and o' as a centre, describe an indefinite arc. The distance these two elements are apart at their lower extremities must be equal to the length of that part of the circumference of the base included between t^h and d^h. Hence with t' as a centre, and a radius equal to t^hd^h, describe an arc cutting the other arc at d'; join d' with o', and we have the developed position of the element A. The other elements are developed in the same way. The greater the number of elements developed, the greater will be the accuracy of the development.

To develop the section cut out by the plane P, find the true lengths of each element from the vertex to the section of, og, etc., and lay them off from o' on the developed positions of those elements respectively $o'f'$, $o'g'$, etc. That portion of the surface included between the development of the base $t'c'x'd't'$, and that of the section $f'g'k'$, etc., will be the development of that part of the cone between the base and the secant plane P.

The tangent line to the point l is developed as explained in Problem 33, the point e being in K^h and T^h, and at a known distance from the points x and l.

103. Fig. 57 shows the construction when the axis of the cone is parallel to both V and H. Fig. 58 shows the development of the cone and the tangent line.

DOUBLE CURVED SURFACES OF REVOLUTION.

104. The general statement for the intersection of a double curved surface of revolution with any plane is as follows: Pass a series of auxiliary planes through the solid perpendicular to its axis. These planes cut circles from the solid and lines from the secant plane, the intersections of which give points on the curve.

105. Problem 36. — *To find the intersection of a double curved surface of revolution with an oblique plane, and to draw a tangent to the curve at any point.*

Let the double curved surface of revolution be a torus given as in Fig. 61, and let P be the cutting plane. Assume the auxiliary planes R, S, X, etc., perpendicular to the axis. The plane Y cuts from the torus two circles, horizontally projected in A^h and B^h, and vertically projected in VY, it cuts from the plane P the horizontal line Y; d, f, l, and n, the intersections of these circles and line, are four points on the curve. Similarly the plane R gives the point a; the plane X gives c, g, k, and o· the plane S gives b and p; and the plane Z gives e and m.

The tangent line T to any point as n, and the true size of the section, etc., are shown in the figure, and need no further explanation.

SOLIDS BOUNDED BY PLANE SURFACES.

106. In finding the intersection of prisms, pyramids, or any solid bounded by plane surfaces with an oblique plane. pass auxiliary planes through the edges of the solid, and perpendicular to V or H; or, in other words, find where each edge of the solid pierces the secant plane by Art. 56.

107. PROBLEM 37. — *To find the intersection of a prism with an oblique plane.*

Let the prism be pentagonal, and given as in Fig. 62, and let P be the cutting plane. The edges A, B, C, etc., pierce the plane P in the points a, b, c, etc.; joining these points we get the two projections of the intersection. Its true size is shown at $a_{\prime}b_{\prime}c_{\prime}d_{\prime}e_{\prime}$.

The development may be found directly here, without getting a right section as in the cylinder, by revolving each face about its edge resting on H into H (Art. 47). $abgf$ revolves to f^hg^hon; $bckg$ to g^hk^hrp, etc.

108. We have seen how, if we have a solid given by its projections, we can obtain its covering or development; hence, if we have the covering given, we are able to construct the projections of the solid.

109. PROBLEM 38. — *To construct the projections of a solid from its covering, and find the intersection of this solid with an oblique plane.*

Let the covering consist of eight regular hexagons and six squares, disposed as shown in Fig. 63. Let one of the hexagons be considered as the base, and let it be placed parallel to H, so that one of its sides, ab, is perpendicular to V. Revolve the hexagon A and the square K until the sides a^hg' and a^hk' coincide, the extremities g' and k' of these sides will meet at a point in space, of which the horizontal projection is evidently x^h; revolve square M until its side f^ht' coincides with the side o^hf^h of the revolved hexagon A, the extremities meet at a point in space of which the horizontal projection is o^h. It is evident that the horizontal projections of the two squares K and M, in their revolved positions, are $a^hb^hs^hx^h$ and $e^hf^ho^hp^h$. To complete the horizontal projection of the hexagon A, we draw through x^h the line x^hw^h parallel and equal to the line f^ho^h; through w^h the line w^hn^h equal and parallel to a^hf^h; and finally join n^h and o^h, and n^ho^h will be equal and parallel to a^hx^h (Art. 13). The vertical projection of the base is the line $a^vf^ve^v$ at any required height above H. The vertical projection of the square K, in its revolved position, is the line a^vx^v, found as follows: Since the plane of the square is perpendicular to V, the point k' moves in revolution in the arc of a vertical circle, with the point a as a centre; hence, when the point has gone to x^h, its vertical projection must be at x^v, where a perpendicular through x^h intersects this circle. Since the squares K and M are inclined at equal angles to H, their top points must be the same height above H in their revolved positions; hence o^v and p^v must be found in a horizontal line through x^v, and vertically above o^h and p^h. Hence the vertical projection of the square M is $e^vf^vo^vp^v$. The vertical projection of the hexagon A is $a^vf^vo^vn^vw^vx^v$, the points a^v, f^v, o^v, and x^v being coincident with those of the squares already found. The point w^v is obtained by dropping a perpendicular from w^h, and producing it until it meets a line drawn through x^v parallel to f^vo^v; or, in other words, make x^vw^v equal and parallel to f^vo^v; make o^vn^v equal and parallel to a^vx^v; and n^vw^v equal and parallel to a^vf^v.

The other hexagons and squares are revolved after the same principle, and the completed projections are obtained as shown in the figure.

To find its intersection with the plane P. — Find where the edges of the solid pierce the plane by Art. 56; some of them will and some will not; hence assume any one that you *think* will pierce it, and see if it does by actual construction. If it does not, try another, and so on. The edge *ep* pierces the plane P at the point 6; *fo* at 5; *wn* at 4, etc.; giving 1 2 3 4 5 6 7 8 as the section. Its true size is found as usual.

The lines $1_{\prime}2_{\prime}$, $2_{\prime}3_{\prime}$, $3_{\prime}4_{\prime}$, $4_{\prime}5_{\prime}$, etc., are the developed positions of the lines cut from the several faces by the secant plane.

CHAPTER V.

INTERSECTION OF SOLIDS.

110. To find the curve of intersection of any two solids whatever, pass a series of auxiliary surfaces (usually planes) through the two solids. They will cut lines (straight or curved) from each solid, the intersections of which are points of the required curve.

While the auxiliary planes may be taken in any position, yet, for convenience, they should be chosen so as to cut the simplest curves from the solid, — *i.e.*, straight lines or circles.

111. The following is a list of special cases showing what auxiliary planes should be taken to cut the simplest curves: —

First. Two cylinders with axes oblique to each other; choose auxiliary planes parallel to both axes. Each plane cuts elements from each solid.

Second. Two cones with axes perpendicular or oblique to each other; choose auxiliary planes passing through a line containing the vertex of each cone. Each plane cuts elements from each solid.

Third. Cylinder and cone with axes oblique to each other; choose auxiliary planes passing through vertex of cone and parallel to axis of cylinder. Each plane cuts elements from each solid.

Fourth. In case of the intersection of cylinders, cones, double curved surfaces of revolution, prisms, and pyramids (any one with one of the same or different kind) with their axes parallel; choose auxiliary planes perpendicular to the axes. The planes will cut from these solids simple figures (from the cylinder, cone, and double curved surfaces of revolution, circles, and from the prisms and pyramids, polygons similar to their bases). One of the coördinate planes should be taken perpendicular to the axes.

Fifth. In case of two cylinders, cylinder and double curved surface of revolution, cylinder and cone, whose axes are perpendicular to each other; choose auxiliary planes perpendicular to one cylinder in first case, to axis of double curved surface of revolution in second case, to axis of cone in third case. Coördinate planes should be taken perpendicular to the axes respectively.

Sixth. In case of two double curved surfaces of revolution with axes oblique and intersecting, pass a series of auxiliary spheres, whose centres are at the intersection of the axes, through the two solids.

A prism can be substituted in place of the cylinder, and pyramid in place of the cone, in any of the above cases.

112. The tangent line to the curve of intersection at any point is the line of intersection of the planes tangent to each solid at that point.

113. Problem 39. — *To find the curve of intersection of two cylinders whose axes are oblique to each other, and to draw a line tangent to any point in the curve.*

Let the cylinders be given as in Fig. 64. This comes under the first special case (Art. 111), and the auxiliary planes are taken parallel to the axis of each cylinder. To find the horizontal trace of such a plane (the vertical trace is not needed) we have simply to pass a plane through one axis and parallel to the other (Art. 63). HS is the horizontal trace of such a plane. HR, HX, etc., drawn parallel to HS will be the horizontal traces of planes which are parallel to both axes. Each of these planes cuts elements from each cylinder, the points of intersection of which are points of the required curve.

The plane S cuts the two elements A and B from the left cylinder, and C and D from the right cylinder; these elements intersect at the points 2, 8, 10, and 16, which are four of the required points. The plane R being tangent to one of the cylinders only gives two points, 1 and 9. The plane X gives the four points 3, 7, 11, and 15. By repeating the process any number of points can be found.

After finding a number of points on the curve, crossing and recrossing as it is liable to do, unless some system of numbering is followed, it is very difficult to connect the points so as to form the correct curve. An examination of the system used in figures 64, 65, and 68 will suffice without explanation; the same method is used whether the solids are two cylinders, two cones, or cylinder and cone.

To determine whether there will be one or two curves. — Draw the two extreme planes R and Z; each of them will be tangent to one (but not the same) cylinder, and secant to the other; hence there will be but one curve.

If the two extreme planes were tangent to the same body, and secant to the other, it is evident that the first body would pass through the second, and there would be two separate curves.

To determine what points on the curve are visible. — A point on the curve is visible in plan or elevation, if it is the intersection of two elements, *both* of which are visible in plan or elevation. Thus the point 9, being at the intersection of two elements which are visible in plan, must be on a visible portion of the curve in plan; these elements being invisible in elevation, the curve at that point will be invisible in elevation. The two elements whose intersection gives the point 5 are both visible in elevation, but only one is visible in plan, therefore the point is on a portion of the curve which is visible in elevation, but not in plan.

To draw a tangent line to any point of the curve. — The tangent line to any point of the curve is the line of intersection of two planes tangent to the solids at that point. HT is the horizontal trace of the plane tangent to the left cylinder at the point 14, and HQ that of the plane tangent to the right cylinder at the same point; T, the line of intersection of these two planes, is the tangent line required.

114. PROBLEM 40. — *To find the curve of intersection, etc., of two cones with axes oblique to each other.*

Let the cones be given as in Fig. 65. This comes under the second special case (Art. 111), and the auxiliary planes are passed through a line containing the vertex of each cone.

The line A passes through each vertex, and a^h is its horizontal trace; hence any line, as HX, HY, etc., drawn through a^h, will be the horizontal trace of a plane which will pass through the vertex of each cone, and, consequently, will cut elements from them. HY cuts the elements B and C from the left cone, and

D and E from the right cone; these elements intersect at the points 2, 4, 6, and 8, which will be points on the required curve. Any number of points may be found in the same way.

Here, as in the case of the cylinder in the last problem, the two extreme auxiliary planes X and Z are each tangent to one (but not the same) body and secant to the other; hence there will be but one curve.

In this case one of the cones is supposed to have been removed, consequently those portions of the curve will be visible if they are on visible portions of the remaining cone.

T is the tangent to the point *n*, and is found exactly as in the last problem.

115. PROBLEM 41. — *To find the curve of intersection, etc., of two cones when their axes are perpendicular to each other.*

Let the cones be given as in Fig. 66, the axis of one being perpendicular to H, and of the other parallel to both V and H. In this case we pass the auxiliary planes through a line containing each vertex, as in the last problem, but here it is necessary to use a profile plane and make use of the traces of the auxiliary planes on this plane, as well as their traces on H.

The horizontal trace of the line A, containing the two vertices of the cones, is a^h; its trace on the profile plane is b. HS, HX, etc., are the horizontal traces of planes which will cut elements from the two cones; b^hl^h, b^hc^h, etc., are their traces on the profile plane. Revolving the profile plane about its vertical trace into V, the point b revolves to b_1^v, and the traces of the auxiliary planes on the profile plane are shown at VS′, VX′, etc. The plane S cuts the element D from the cone whose axis is horizontal, and B and C from the upright cone; the intersections of these elements *d* and *e* are two points of the curve. The plane Z gives the points *m*, *n*, *r*, and *t*. Any number of points can be found in the same way.

Since the axes of the two cones intersect, the curve will be the same on each side of the central plane Z; hence, since the plane S is tangent to one cone and secant to the other, there will be two separate curves.

T is the tangent line to a point on the curve, being the intersection of the plane Q, which is tangent to the cone whose axis is horizontal, and the plane T, tangent to the other cone at the common point.

Fig. 67 shows the development of the upright cone, together with the holes in which the horizontal cone penetrated it, and the tangent line.

116. Problem 42. — *To find the curve of intersection, etc., of a cylinder and cone with axes oblique.*

Let the bodies be given as in Fig. 68. This comes under the third case, and the auxiliary planes are passed through the vertex of the cone parallel to the axis of the cylinder. To do this, draw the line A through the vertex of the cone, and parallel to the axis of the cylinder; a^h is its horizontal trace; any line, as HS, HX, etc., drawn through this point will be the horizontal trace of such a plane.

The plane S cuts the element B from the cone, and the two elements, C and D, from the cylinder; the intersections of these elements at 1 and 7 are two points of the required curve. Any number of other points can be found in the same way.

117. Problem 43. — *To find the curve of intersection of a sphere and hexagonal prism.*

Let them be given as in Fig. 69. This is an illustration of the fourth special case, of which there can be a large number.

The horizontal plane X cuts the circle X from the sphere, and a hexagon from the prism; these intersect at the point n. Other points are found in the same manner.

118. Problem 44. — *To find the curve of intersection of a cylinder and torus with axes perpendicular.*

Let them be given as in Fig. 70. This is an illustration of the fifth special case. Here planes should be taken perpendicular to the axis of the torus.

The plane Y cuts from the torus the two circles A and B, and from the cylinder the two elements C and D. These circles and elements intersect in the points a, $a_{\prime}$, $a_{\prime\prime}$, $a_{\prime\prime\prime}$, b, $b_{\prime}$, all of which are points on the required curve. Other points are found in the same manner.

119. In cases where the bodies are bounded by planes, and do not rest with their bases on either of the coördinate planes, it is easier to make use of the method described in Art. 58 than to find the traces of the planes containing the separate faces of the solid.

120. PROBLEM 45. — *To find the intersection of a triangular prism with a triangular pyramid.*

The points *e*, *f*, and *g*, Fig. 71, where the elements A, B, and C of the prism penetrate the face *abc* of the pyramid, are found by Art. 58; joining these points, we have *efg* as the intersection of the prism with the face *abc* of the pyramid. The three elements of the prism do not leave the pyramid in the same face; hence find the points *m* and *n* where the elements A and C leave the face *acd*; then find the point *r* where the element B would leave the same face if that face were considered of indefinite extent; joining *m*, *n*, and *r*, we have the intersection of the prism with the face *acd* when extended sufficiently. That portion of the triangle *mnts* which actually falls on the limited face *acd*, will be a part of the intersection required; the element B actually leaves the pyramid at the point *o* in the face *bcd*; joining this point with *s* and *t*, the intersection is completed.

Fig. 72 shows the development of the surface of the pyramid with the intersections traced on it.

121. PROBLEM 46. — *To construct the projections of a right hexagonal prism and a right hexagonal pyramid, and find their intersection when they occupy a certain fixed position relative to each other and to the coördinate planes.*

Let the lowest corner of the prism rest on H; the long diameter of the base of a prism is 2½ inches; its height is 3 inches; two of its faces are in planes perpendicular to H; its axis makes an angle of 60° with H and 25° with V. The long diameter of the base of the pyramid is 2 inches; its height is 3 inches; its axis slopes downward, forward, and to the left, making angle of 45° with H, and whose horizontal projection makes angle of 15° with GL. A point in the axis of the prism 2 inches from the base is ½ inch, perpendicular distance, behind a point in the axis of the pyramid which is 1 inch from the base of the pyramid.

See Fig. 73 for construction. If the axis of prism makes an angle of 60° with H and 25° with V, the plane of the base makes an angle of 30° with H and 65° with V. VP and HP, the traces of this plane, are found by Art. 70. The projections of the prism are then found by Art. 72.

Next it is necessary to locate the axis of the pyramid. To do this we use the converse of the principles in Art. 66, which explains how to find the perpendicular distance between two lines when their projections are given; here the projections of one line, the direction of the projections of the other, and the perpendicular distance between them are given, to find the projections of the second. d, the point in the axis of the prism, 2 inches from the base, is found by laying off on the revolved position of the axis $a_{,}^{h}b_{,}^{h}$, the distance $a_{,}^{h}d_{,}^{h}$, equal to 2 inches, and in counter revolution $d_{,}^{h}$ goes to d^{h}. Now, through this point d, draw the line A parallel to the axis of the pyramid by the converse of the principles in Art. 36. $A_{,}^{v}$ is drawn through d^{v}, making angle of 45° with GL. $A_{,}^{h}$ must, of course, be parallel to GL. From d^{v} lay off on $A_{,}^{v}$, a distance $d^{v}e_{,}^{v}$, equal to 2 inches, and $d^{v}n_{,}^{v}$, equal to 1 inch; in counter revolution $A_{,}^{h}$ moves through angle of 15° to A^{h}, $A_{,}^{v}$ moves to A^{v}, $e_{,}^{v}$ to e^{v}, $e_{,}^{h}$ to e^{h}, $n_{,}^{v}$ to n^{v}, and ne is a line through d equal in length and parallel to the axis of the pyramid. Through the line A and the axis of the prism pass a plane. VQ and HQ are the traces of this plane. From the point d draw the perpendicular do of indefinite length; revolve this indefinite line until it is parallel to V, and lay off $d^{v}o_{,}^{v}$ equal to $\frac{1}{2}$ inch on the revolved position; in counter revolution $o_{,}^{v}$ moves to o^{v}, and o^{h} is found vertically below it on the horizontal projection of the indefinite perpendicular. Through the point o draw the line fg equal and parallel to en, and this line will be the axis of the pyramid correctly located relative to the axis of the prism.

Through the point g pass the plane R perpendicular to the axis fg of the pyramid. This will contain the base of the pyramid, and the projections of the pyramid can then be found the same as the prism by Art. 72.

The intersection of the two bodies is obtained by finding where each edge of one penetrates the faces of the other, if at all, by Art. 58. the same as in Art. 120.

122. PROBLEM 47. — *To find the curve of intersection of two double curved surfaces of revolution with their axes intersecting.*

Let the two surfaces be an ellipsoid and a sphere, given as in Fig. 74, their axes intersecting at *o*.

Intersect these two surfaces by a series of auxiliary spheres whose centres are at *o*; each sphere will cut circles out of the surfaces, the intersections of which will be points on the required curve. The sphere A cuts out of the ellipsoid the circle C, and out of the sphere the circle D, their intersections *e* and e_1 are two points on the curve. Other points can be found in the same way.

CHAPTER VI.

Miscellaneous Problems.

123. Problem 48. — *To find the right section of a cylinder, making a certain known angle with the horizon, which will exactly mitre with a horizontal cylinder whose axis is at right angles to it.*

Let the horizontal cylinder be circular, and with its axis perpendicular to V, Fig. 75; the other cylinder makes an angle β with H, and is parallel to V. The section of each cylinder on the mitre plane Q is the same.

The plane P will cut a right section from the slanting cylinder, whose vertical projection will be in the line VP, and whose horizontal projection can be found as follows: The plane R is tangent to both cylinders along the elements A and B, which must intersect at the point a in the plane Q. In this way A^h is determined, and consequently b^h, a point on the horizontal projection of the curve. In the same way the plane T gives the two points d and $d_{\prime}$. Other points are found in the same way.

The true size of the section $dbd_{\prime}$ is easily found.

124. Problem 49. — *To find the right section of the raking moulding which will mitre with a gutter, the pitch of the roof and the section of the gutter being given.*

This is an application of the principles of the preceding article.

Let the section of the gutter be given as in Fig. 76, and the pitch of the roof equal the angle β.

The points and planes have been lettered the same as in Fig. 75, so that the explanation for that figure is equally applicable for this, except that in this figure the true size of the section is shown as revolved into a plane *parallel* to V instead of *into* H. It is not necessary that the back part of the moulding be made to follow the inside of the gutter, as it is invisible; hence it is made out of an ordinary plank of any convenient thickness, and the curve *exdb* is all that it is necessary to find.

It is well to note that, when the plane Q makes an angle of 45° with V, as is usual in such work, the distances d^hk, x^hc, etc., are equal to *or*, *st*, etc., respectively; hence the true size of the section can be obtained directly from the elevation without the use of the plan, as follows: Draw any number of lines, as VR, VS, etc., making angle β with the horizon; draw any line, as VP, at right angles to these lines; from the points of intersection d^v, x^v, etc., of the line VP with VT, VX, etc., lay off along these lines $d^vd_{\prime}^v$, $x^vx_{\prime}^v$, etc., equal to *or*, *st*, etc., the distance from the points where VT, VX, etc., intersect the curve of the gutter to any vertical line as C. A curve drawn through the points $d_{\prime}^v$, $x_{\prime}^v$, etc., will be the curve of the moulding.

125. Problem 50. — *Given the right section of the sloping part of the side wall to a flight of steps, to find the right section of the lower end of this wall, at right angles to the given section.*

Let $ed_{\prime}^vx_{\prime}^vb$, Fig. 77, be half the given section, and β the angle of the slope of the wall.

This problem is simply the converse of the preceding one, and needs no further explanation.

126. Problem 51. — *To find the covering of an eight-sided dome as given in Fig.* 78.

The right section of the portion *oab* is the curve $a^vt^vs^v$, etc. This right section unrolls in the straight line $e^ht_{\prime}s_{\prime}$, etc., $e^ht_{\prime}$, $t_{\prime}s_{\prime}$, etc., being equal to a^vt^v, t^vs^v, etc., respectively.

The right section of the portion *obc* is shown at b^ho', and it unrolls in the straight line $b^ho_{\prime\prime}^h$; $b^ho_{\prime\prime}^h$ being equal in length to the arc b^ho'.

The right section of the portion *ocd* is the ellipse shown at n^ho'', which unrolls in the straight line $n^ho_{\prime\prime\prime}$, equal in length to the arc n^ho''.

Since most of the remaining problems are simply applications of principles already familiar, I have thought it best to give the construction without the description in some cases; a general description without the construction in others: and still others without either construction or description, in order that the student may accustom himself to apply these principles to the solution of practical problems as they may arise.

127. Problem 52. — *Given the elevation of the outline of a twelve-sided vase, as in Fig.* 79, *to find the elevation of the edges A, B, C, and D, thus completing the elevation of the object; also the development of one of the sides.*

128. Problem 53. — *To strike the pattern of the sheet metal used in making a bath-tub.*

Figs. 80, 81, and 82 show the construction for three different styles of tubs.

129. Problem 54. — *To find the intersection of the steam dome and boiler of a locomotive; to develop the steam dome; and also to develop the taper sheet.*

Fig. 83 shows the two elevations of a portion of a locomotive, and the intersection of the steam dome and boiler.

Fig. 84 is the development of the steam dome, and Fig. 85 is the development of the taper sheet without the lap.

130. Problem 55. — *To find the position of a guide pulley, and its shaft, to take a belt in either direction from one pulley to another whose shafts are at right angles to each other.*

Let A and B, Fig. 86, be the given pulleys. Assume any point *c* in the line *ab* (as the guide pulley may be placed at any convenient point between the two given pulleys); from this point draw the lines *cm* and *ce* tangent to the pulleys B and A respectively. The plane P of these two lines *cm* and *ce* will be the plane of the guide pulley.

The ground line may be taken in any convenient position perpendicular to the given shafts.

To find the projections of the guide pulley, its diameter being known, revolve the plane P into the horizontal coördinate plane; *cm* revolves to $c_{,}^{h}m_{,}^{h}$, and *ce* to $c_{,}^{h}e_{,}^{h}$; $C_{,}^{h}$ is then drawn equal to the actual size of the guide pulley. In counter revolution its centre is found at (o^{v}, o^{h}), and the circle $C_{,}^{h}$ is found at (C^{v}, C^{h}), as explained in Art. 72.

The shaft being perpendicular to the plane of the pulley, its projections are respectively perpendicular to the traces VP and HP, and they must pass through the centre (o^v, o^h); hence through the point o draw the indefinite lines o^vn^v perpendicular to VP, and o^hn^h perpendicular to HP, and the direction of the shaft will be determined.

To show a definite length of the shaft, revolve the line rs until it is parallel to V; then, in its revolved position, lay off $o_{\prime}^h s_{\prime}^h$ and $o_{\prime}^h r_{\prime}^h$, each equal to half the length of the shaft; in counter revolutions $r_{\prime}$ goes to r, $s_{\prime}$ to s, and rs is the length required.

131. Problem 56. — *Given the longitudinal section and plan, or end view, of a connecting rod of varying cross section, to complete the drawing of the elevation.*

Let the section and plan be given as in Fig. 87. The lower end from A to B is cylindrical, from C to D is octagonal, from E to F is rectangular, between B and C the cylinder expands regularly in the arc of a circle from the cylindrical to the octagonal section, and between D and E it expands in an elliptical curve from the octagonal to the rectangular section.

132. Problem 57. — *Given the section and plan of a portion of a rocker arm, as in Fig.* 88, *to complete the elevation.*

There is but one curve to be found, and that is simply the intersection of a cylinder and torus, whose axes are parallel to each other. The cylinder and torus are shown in the figure by dotted lines.

133. Problem 58. — *To find the development of the conical portion B of the double elbow shown in Fig.* 89, *the right sections of the portions A and C being circular.*

134. Problem 59. — *To find the section on AB of the screw, as given in Fig.* 90; *also, find section on CD.*

135. Problem 60. — *To construct the shadow of a sphere on a horizontal plane; also, find its line of shade.*

Pass a plane P, Fig. 91, perpendicular to H, through the ray of light which passes through the centre of the sphere. Revolve this plane into H. The great circle which it cuts from the sphere revolves to the circle of which o_1^h is the centre; the ray of light through the centre revolves to $o_1^h a^h$; $c_1^h b^h$ and $d_1^h o^h$, parallel to $o_1^h a^h$, are the revolved positions of the two rays of light lying in the plane P, and tangent to the sphere; o^h and b^h, where these two lines intersect HP, are the two extremities of the long diameter of the ellipse which is the shadow of the sphere; the short diameter of this ellipse $e^h x^h$ will, of course, be equal to the diameter of the sphere.

To get the line of shade, revolve the plane P back to its original position; the point c_1^h goes to c^h, and d_1^h to d^h; c^h and d^h are two points of the ellipse which constitutes the line of shade.

APPENDIX.

This Appendix is added in order to explain some special cases which are not discussed in the body of the book, in the hope that it will be of assistance to both the instructor and the student.

PROBLEM 3. — *If one end of a line is in a coördinate plane, the true length may be found wholly with the dividers.*

Consider the line *ab*, Fig. 1, the point *a* being in H. Revolving about the H projection into H, the true length is $a^hb_1^h$. This is the hypothenuse of a right angled triangle, whose two sides are a^hb^h and b^vr, *i.e.*, the length of the H projection, and the distance of the V projection (of the end not on H) from GL.

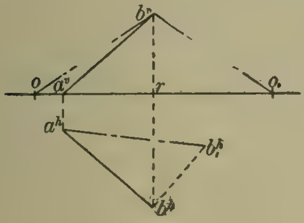


Fig. 1.

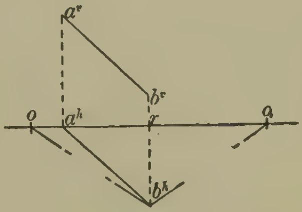


Fig. 2.

Hence to find the true length, take a^hb^h in the dividers; measure from r along GL to o (or o_1); then from o (or o_1) to b^v is the true length.

If one end of the line is on V, the true length is the hypothenuse of a right angled triangle whose two sides are the length of the V projection, and the distance of the H projection (of the end not on V) from GL. Take a^vb^v, Fig. 2, in the dividers; measure from r to o (or o_1); then ob^h (or o_1b^h) is the true length.

THE PROFILE LINE. — Let $a^vb^v - a^hb^h$, Fig. 3, be the given line. Revolve into V by drawing perpendiculars to the V projection. Point a is in front of V the distance of a^h to GL. Point b is behind V the distance of b^h to GL. $a_{\prime}^{v}b_{\prime}^{v}$ is the revolved position. The point r^v, where the revolved position crosses the V projection, is in V; it is, therefore, the V trace of the line, and its H projection, r^h, is in GL.

Consider the point on the line (produced) whose V projection is c^v. This must revolve perpendicular to the V projection and be found at $c_{\prime}^{v}$, on the revolved position (produced). Then $c^vc_{\prime}^{v}$ equals the distance of c^h from GL.

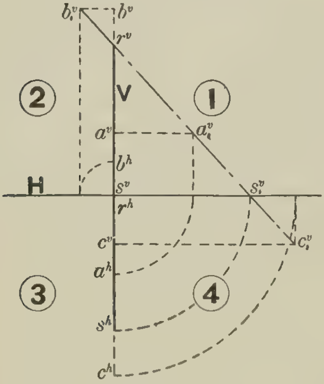


Fig. 3.

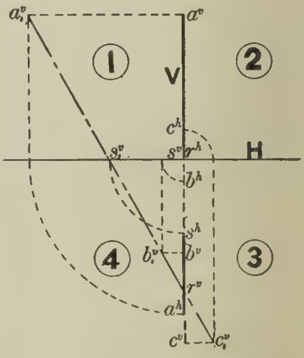


Fig. 4.

Similarly if s^v is the V projection of a point on ab, its H projection is s^h; but if s^v is on GL, point s is in H and is the H trace of the line.

We may consider that the revolved position represents an end view of the coördinate planes; the projection representing V, and GL representing H. The quadrants may be identified by noting that point a is in the first quadrant, point b in the second, and point c in the fourth. $r^vs_{\prime}^{v}$ shows the part of the line which is in the first quadrant; hence r^vs^v and r^hs^h are full lines.

Fig. 4 shows a different line ab revolved into V. r^vr^h is its V trace, s^hs^v is its H trace; point a is in the first quadrant, point b in the fourth, and point c in the third. $a_{\prime}^{v}s_{\prime}^{v}$ shows the part of the line which is in the first quadrant.

To revolve into H, perpendiculars must be drawn to the H projection, and the distances from GL to the V projections laid off. This is done in Fig. 5. $a_{/}^{h}b_{/}^{h}$ is the revolved position; where this crosses the H projection is the H trace; the point on GL, $r_{/}^{h}$, gives the V trace.

The revolved position represents an end view again, only in this case the projection represents H and GL represents V.

Point *a* is in the first quadrant, point *b* in the fourth, and point *c* in the third. $a_{/}^{h}s^{h}$ shows the part of the line which is in the first quadrant.

Fig. 6 shows a different line revolved into H. $s^{h}s^{v}$ is the H trace; $r^{v}r^{h}$ is the V trace; point *a* is in the first quadrant, point *b* in the third, and point *c* in the second. $a_{/}^{h}r_{/}^{h}$ shows the part of the line in the first quadrant.

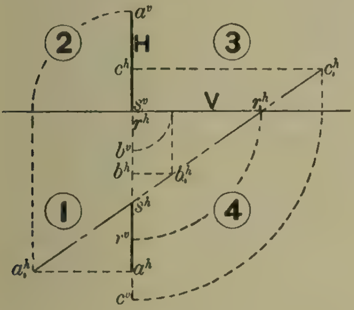


Fig. 5.

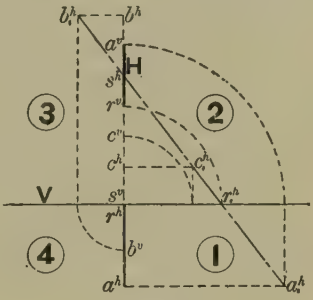


Fig. 6.

In revolving the profile plane into either V or H, the following rules always hold.

To revolve into V, swing the H points to GL; then GL represents H, the projection represents V, and the first quadrant is above GL. To revolve into H, swing the V points to GL; then GL represents V, the projection represents H, and the first quadrant is below GL.

Revolving into either V or H, all arcs (showing revolution or counter revolution) are drawn in two opposite quadrants; viz., the second and fourth. The second quadrant is always above GL; the fourth always below.

Problem 6. — In Fig. 7 the distance $a_{/}^{h}x^{h}$ is the true length of the line $a^{h}x^{h} - a^{v}x^{v}$, which by Problem 3, the point x being on H,

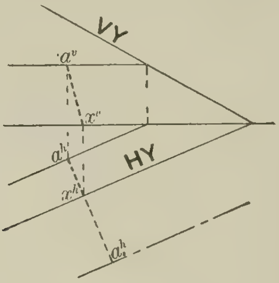


Fig. 7.

is the hypothenuse of a right angled triangle whose two sides are $a^{h}x^{h}$, and the distance from a^v to GL.

Problem 7. — *To bisect the acute angle between lines* A *and* B, *Fig.* 8.

Having found the angle by revolving into H, draw the bisector

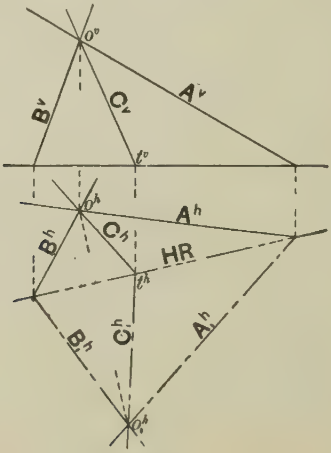


Fig. 8.

$C_{/}^{h}$. $o_{/}^{h}$ revolves back to o^h; t^h, where $C_{/}^{h}$ meets HR, does not move; C^h passes through o^h and t^h. o^h projects to o^v; t^h, being in H, projects to GL at t^v; C^v passes through o^v and t^v.

If $C_{/}^{h}$ does not intersect HR within the limits of the paper, Fig. 9, we may proceed as follows: From any point on HR, as m^h, whose V projection is on GL at m^v, draw an auxiliary line X parallel to one of the given lines, as A. $X_{/}^{h}$ is parallel to $A_{/}^{h}$, X^h is parallel to A^h, X^v is parallel to A^v.

$X_{/}^{h}$ intersects $C_{/}^{h}$ at $w_{/}^{h}$; since w is a point on line X, w^h is on X^h, and w^v on X^v. But w is a point on line C; o is another; hence C^h passes through o^h and w^h, and C^v through o^v and w^v.

The auxiliary line X is taken parallel to one of the given lines for mechanical convenience only. It could have been taken *anywhere* intersecting HR, one of the given lines, and the bisector. In which case the projections of X would have been obtained by counter revolving $n_{/}^{h}$, the point where $X_{/}^{h}$ intersects $B_{/}^{h}$, to $n^h - n^v$ on $B^h - B^v$, and n is one point on X, m is another.

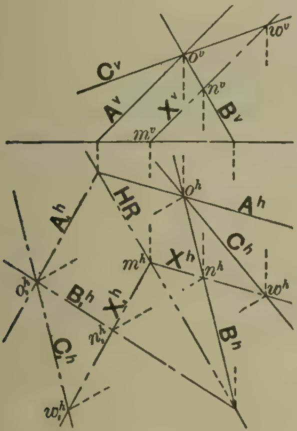


Fig. 9.

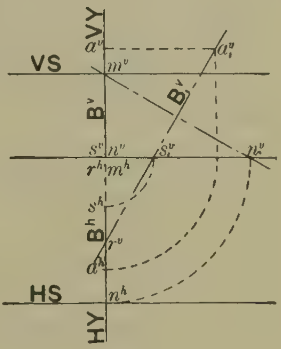


Fig. 10.

PROBLEMS 10, 13, and 18. — *To find the traces of the line* B, *Fig.* 10, *which passes through the point a and is perpendicular to plane* S; *the plane* S *being parallel to* GL.

Draw the profile plane VY, HY. This contains the line B; it also cuts a line, $m^vn^v - m^hn^h$, from the plane S, to which B is perpendicular. Revolving into V, point a goes to $a_{/}^{v}$; the line cut from plane S takes the position $m^vn_{/}^{v}$. $B_{/}^{v}$ passes through $a_{/}^{v}$ perpendicular to $m^vn_{/v}$; its traces (Problem 3) are the points r and s.

PROBLEM 22. — *To find the angle made by two oblique planes.*

Let the given planes be P and Q, Fig. 11. They intersect in the line $A^v - A^h$ (Art. 50). Pass the plane X perpendicular to line A (Art. 29); its H trace, HX, will be sufficient. This cuts a line L from plane P, one point of which is *e*; and a line M from plane Q, one point of which is *g* (Art. 50). Another point on both L and M is *o*, where line A pierces plane X. To find this point, pass the H projecting plane Z through line A, and revolve into H (Art. 37); $A_{/}^{h}$ is the revolved position of A. Plane Z cuts from plane X a line, one point of which is *t* (Art. 50); this line is

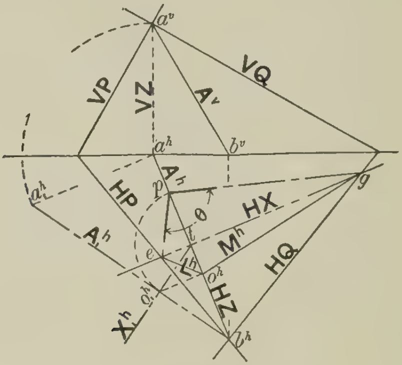


Fig. 11.

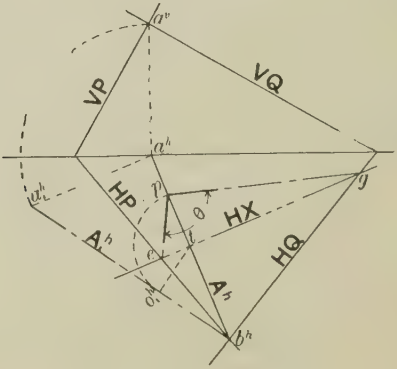


Fig. 12.

perpendicular to A, since plane X is perpendicular to A; hence its revolved position $X_{/}^{h}$ is drawn through t perpendicular to $A_{/}^{h}$, intersecting $A_{/}^{h}$ at $o_{/}^{h}$. $o_{/}^{h}$ revolves back to o^h, and L^h and M^h may now be drawn. The angle between the planes P and Q is the same as the angle between lines L and M. Revolving about HX (Art. 46), o^h goes to p, the distance tp being the true length of the line to^h, which is $to_{/}^{h}$; e and g remain stationary, giving the angle θ.

Since the true size of the angle θ is all that is required, it is not necessary to draw the two projections L^h and M^h; and as only the point a^v on A^v is used, A^v may also be omitted. The solution of the problem with all unnecessary lines omitted is shown in Fig. 12.

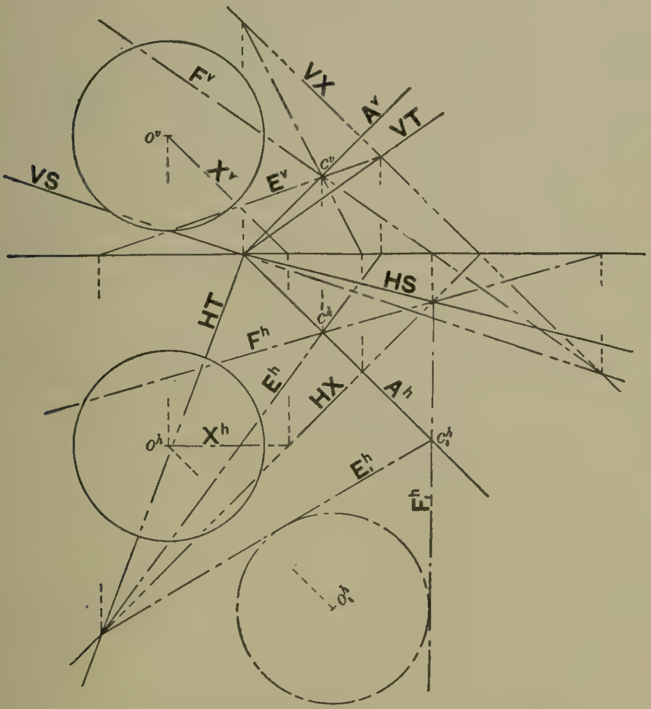


Fig. 13.

Fig. 13 shows the construction described in Problem 31.

PROBLEM 36. — *To find points determined by meridian planes.*

The principal meridian plane Q cuts from the torus the two circles which show in the V projection; it cuts from plane P the line C. C^v intersects the circles in the points k^v, l^v, m^v, and n^v, which are points on the curve.

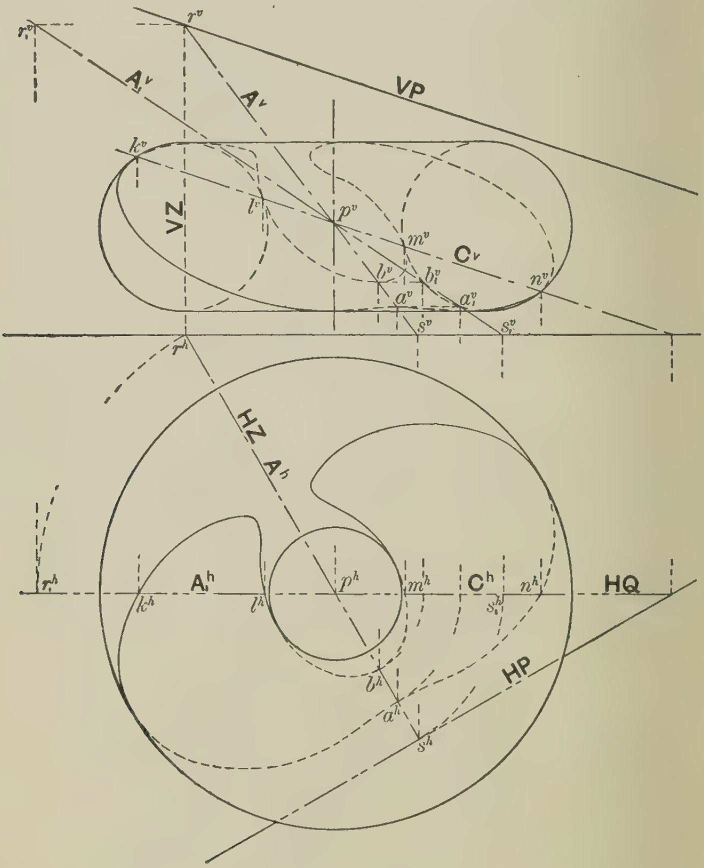


Fig. 14.

The meridian plane Z is taken so as to cut the line of greatest declivity from the plane P (Art. 67); this line is $A^v - A^h$ (Art. 50). Revolve plane Z about the axis of the torus until it coincides with the principal meridian plane Q; the two circles cut by Z from the torus are then projected in V as before. The line A takes the position $A_{/}^{v} - A_{/}^{h}$ (Art. 36), intersecting the circles at the two points $a_{/}^{v}$ and $b_{/}^{v}$, which are the revolved positions of two points a and b on the curve.

This plane Z, which contains the axis of torus and line of greatest declivity of plane, divides the curve into symmetrical parts, determines directly the number of curves, and, in a general way, the shape of the curves to be obtained. When, as in Fig. 14, the revolved position $A_{/}^{v}$ intersects one circle, but not the other, giving two points, there is one curve, extending farther and farther around the torus the nearer $A_{/}^{v}$ approaches the circle it does not intersect. If $A_{/}^{v}$ intersects both circles, giving four points, there are two curves, one inside the other. If, on the contrary, $A_{/}^{v}$ passes between the two circles, intersecting neither, and giving no points, there are two curves, one on each side of the plane Z.

Fig. 1.

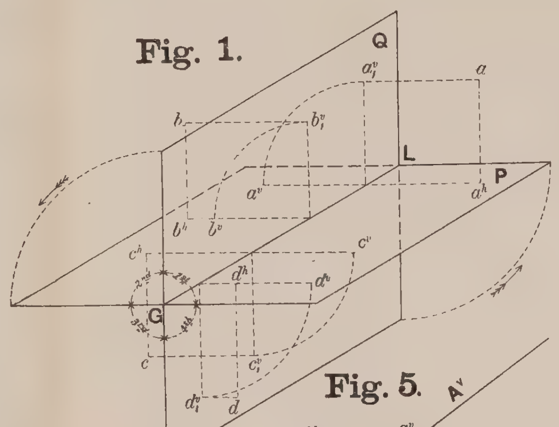


Fig. 2.

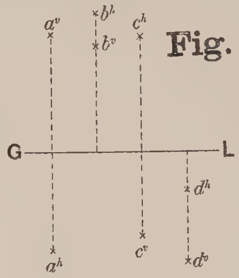


Fig. 3.

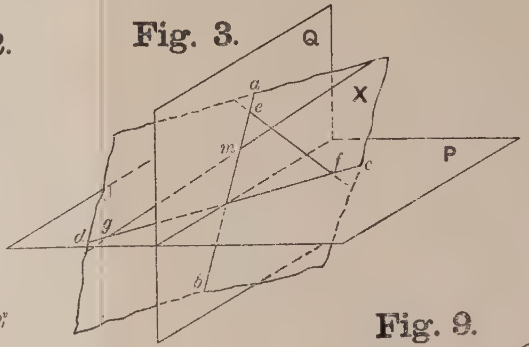


Fig. 4.

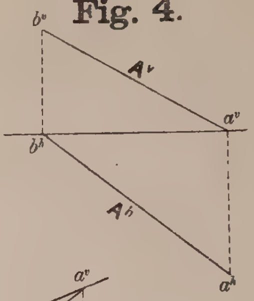


Fig. 5.

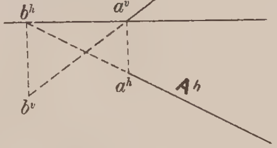


Fig. 6.

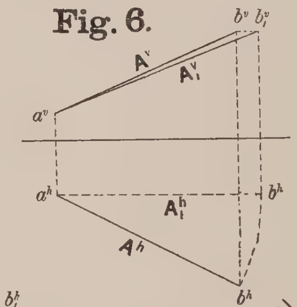


Fig. 9.

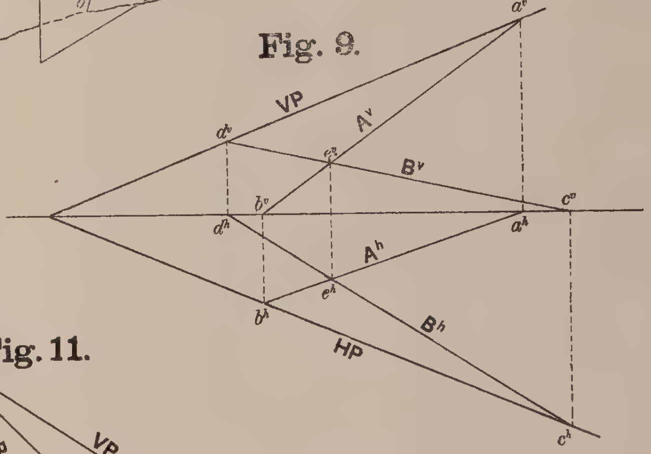


Fig. 7.

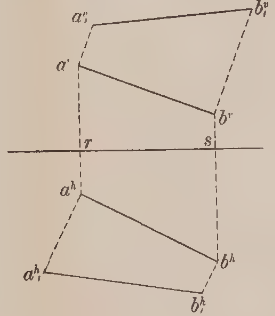


Fig. 8.

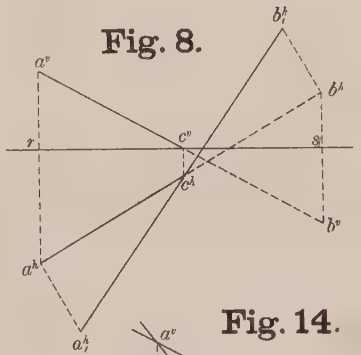


Fig. 11.

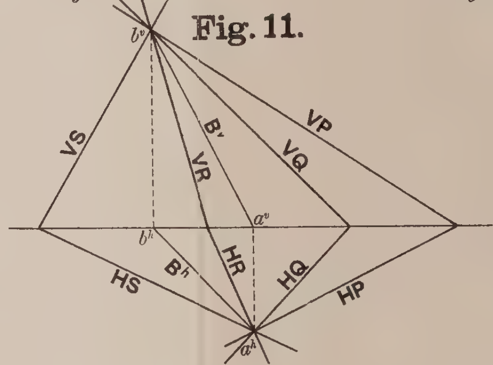


Fig. 10.

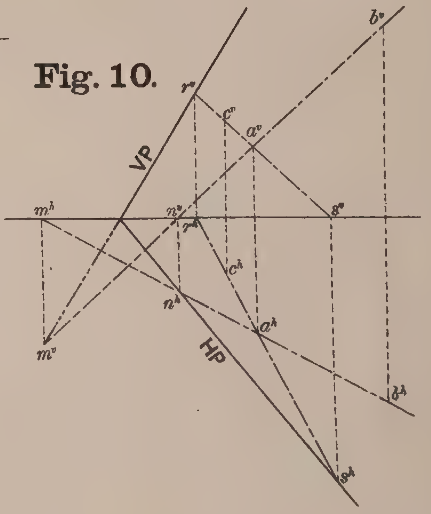


Fig. 12

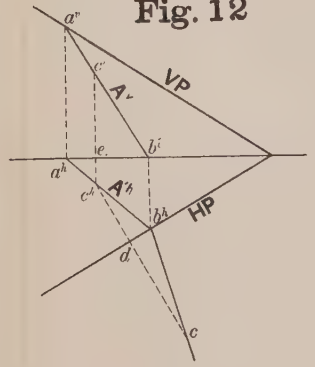
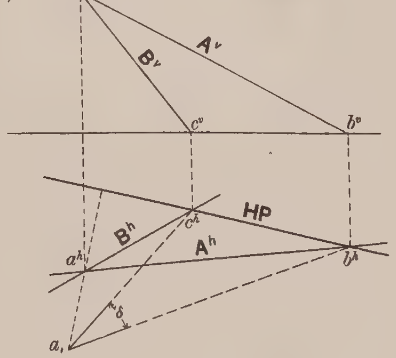


Fig. 14.



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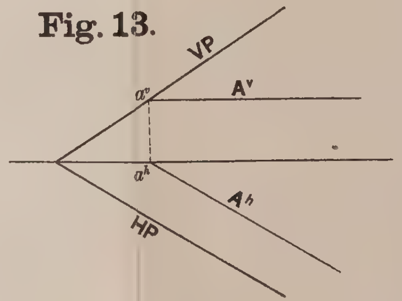

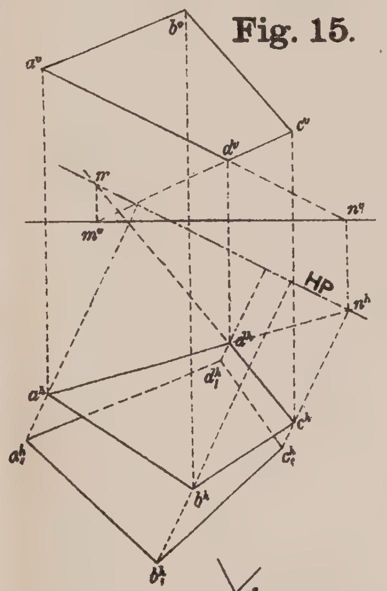
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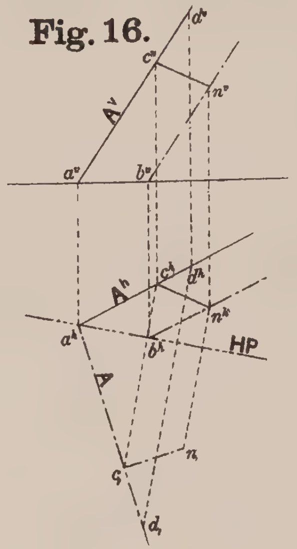
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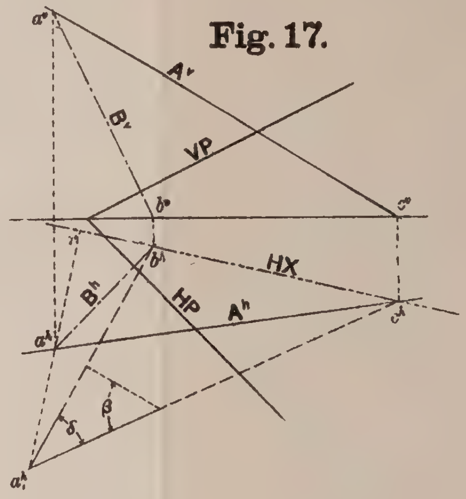
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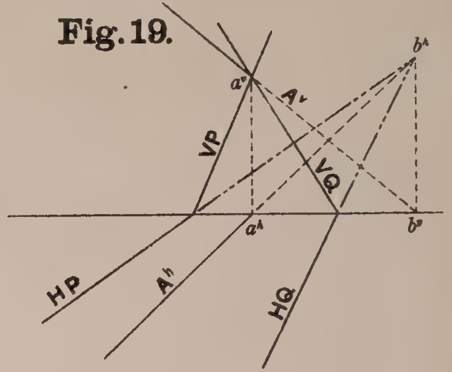
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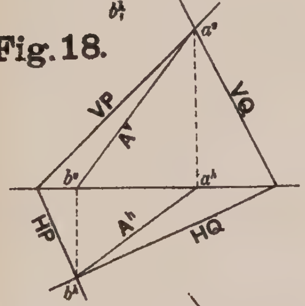
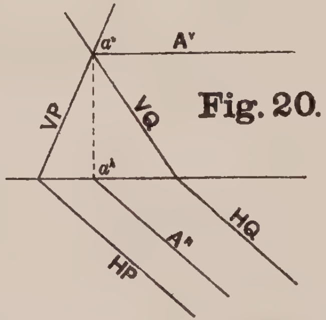
Fig. 18.

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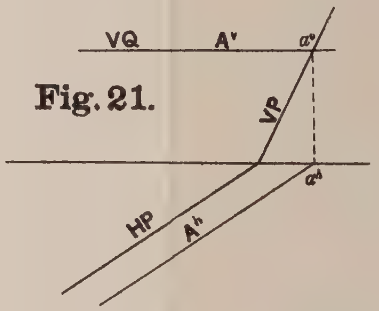
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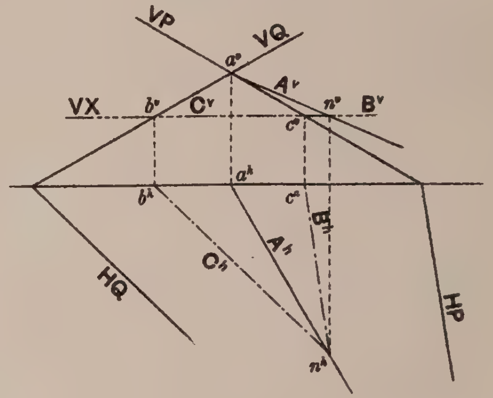
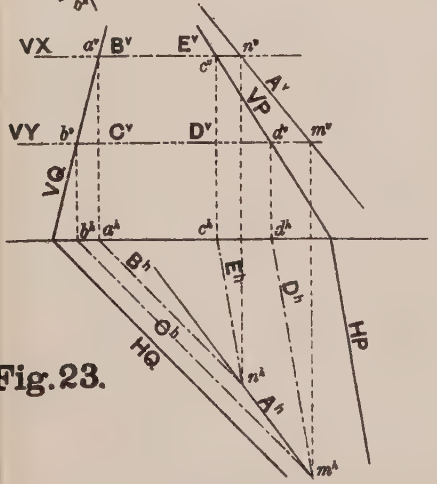
Fig. 22.

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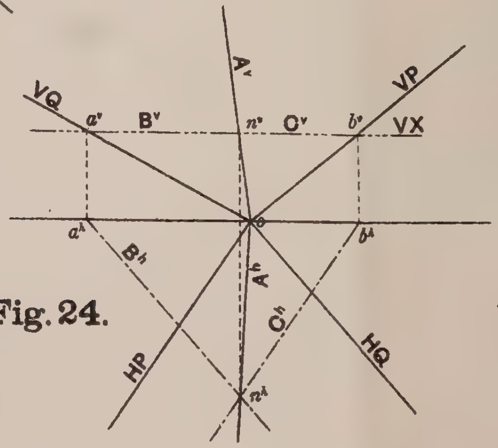
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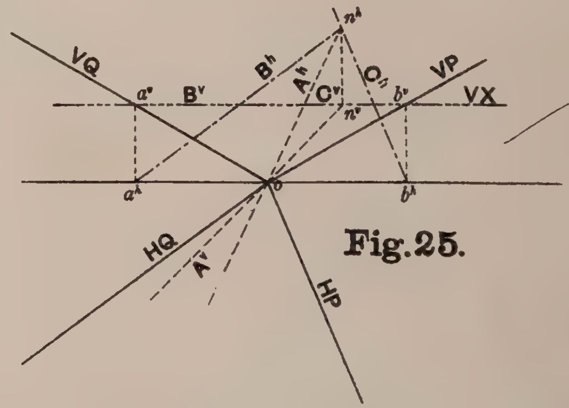
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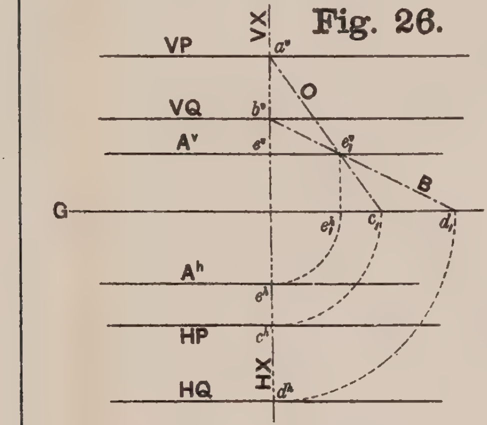
Fig. 26.



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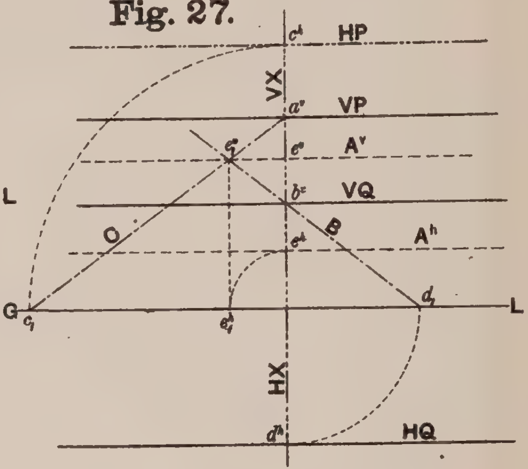

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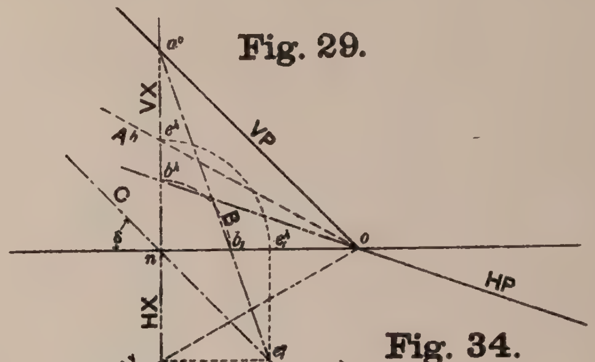

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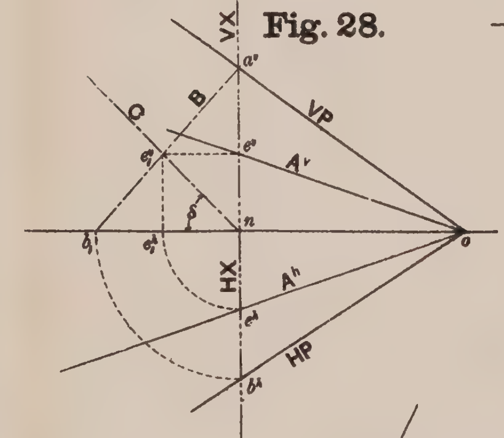

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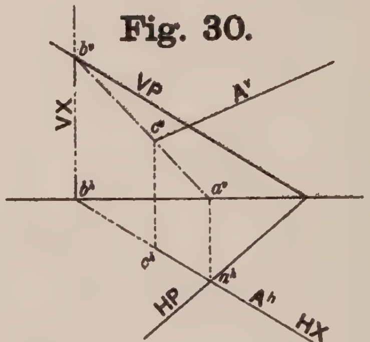

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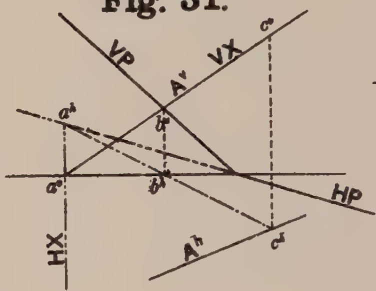

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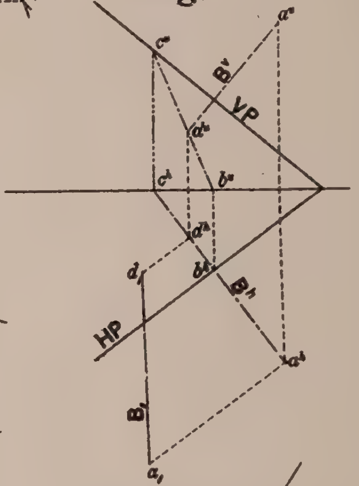


Fig. 32

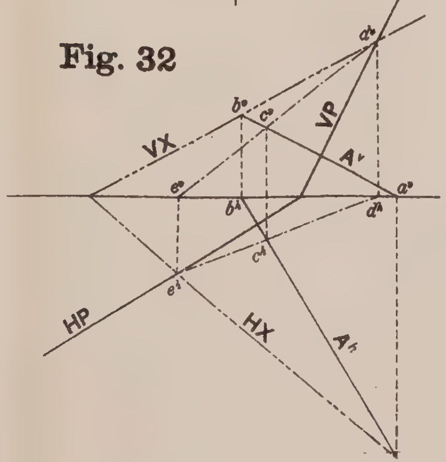
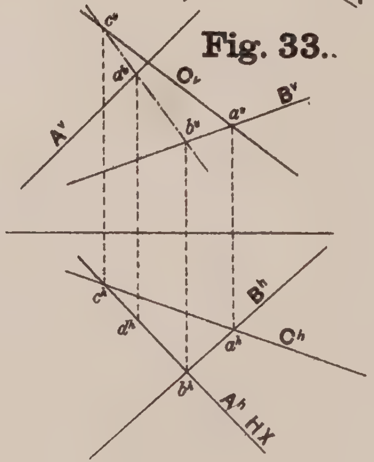


Fig. 33..



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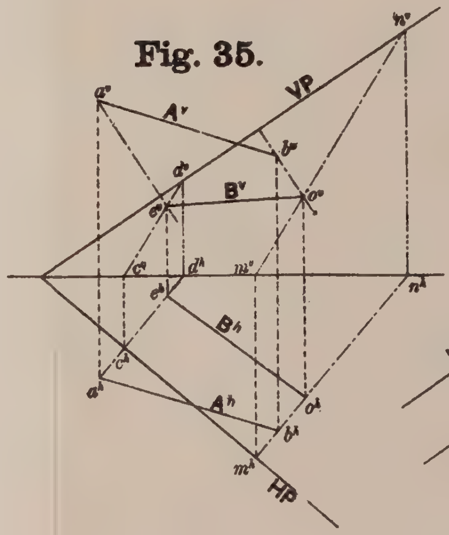

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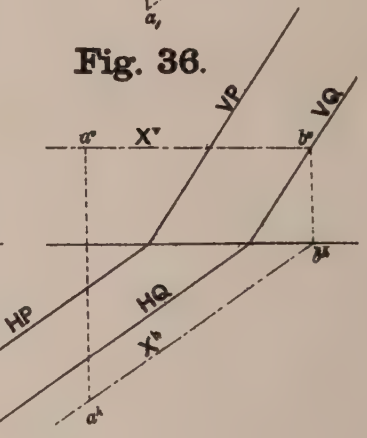

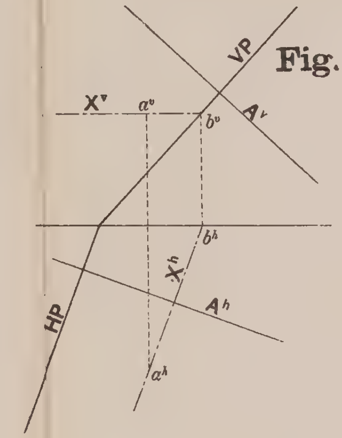

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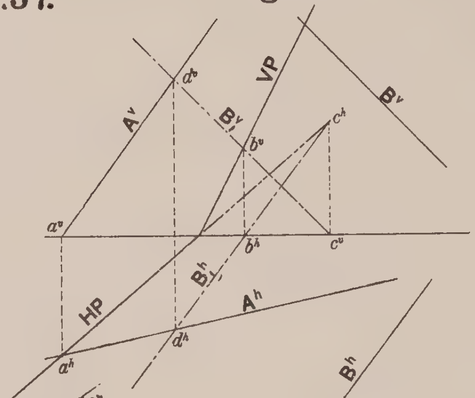

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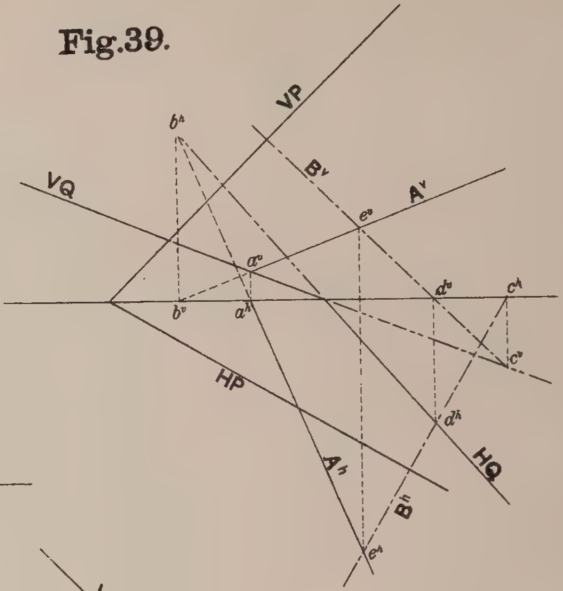

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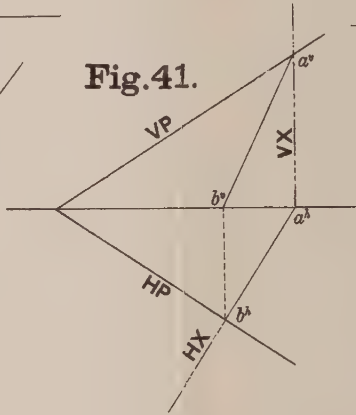

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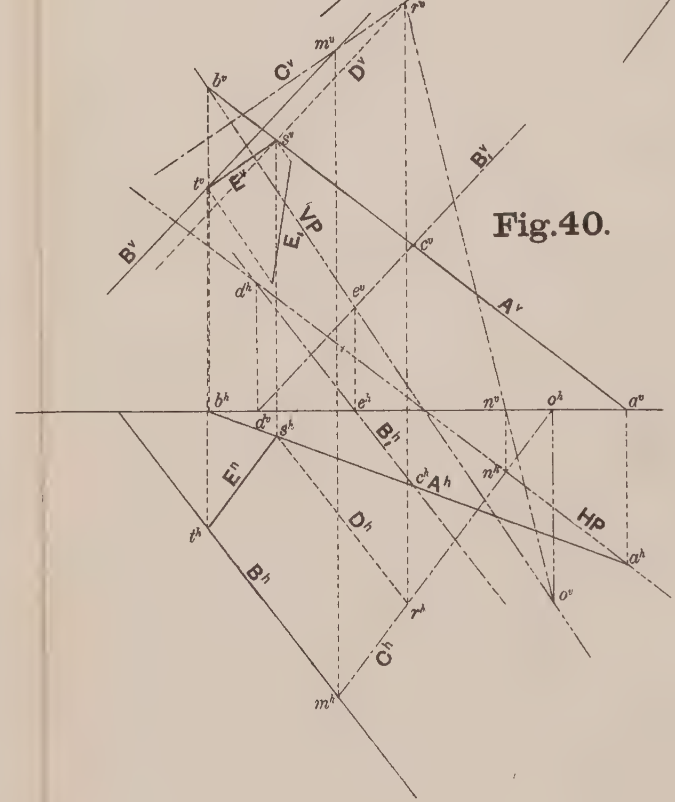

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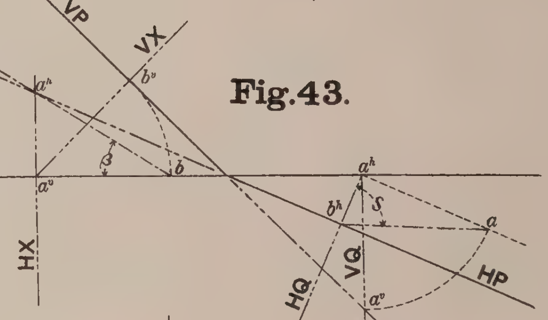

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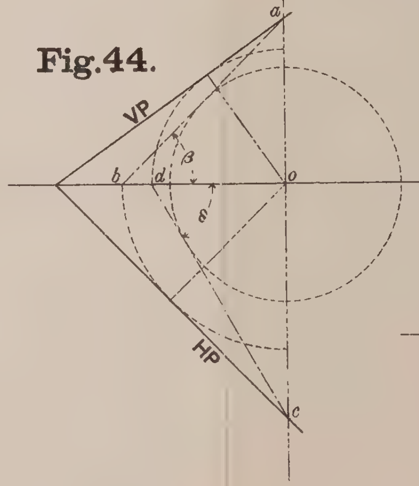

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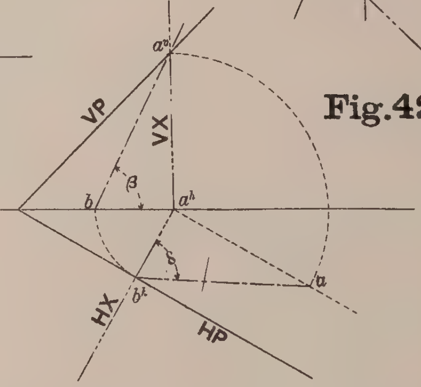
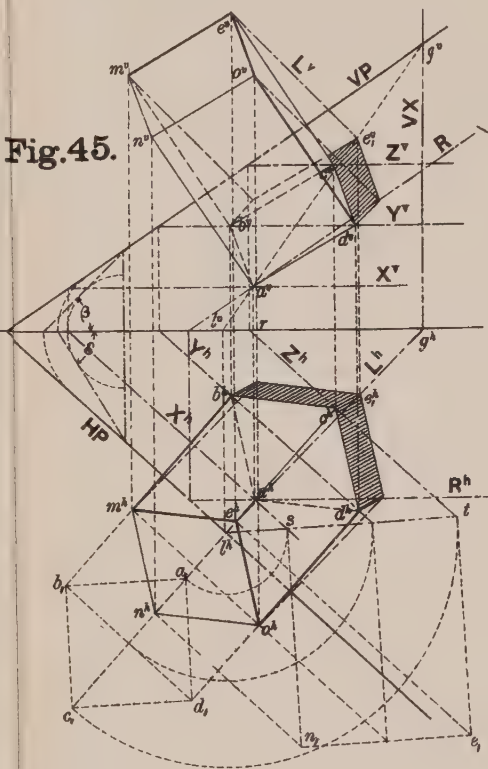


Fig.42.

Fig.45.



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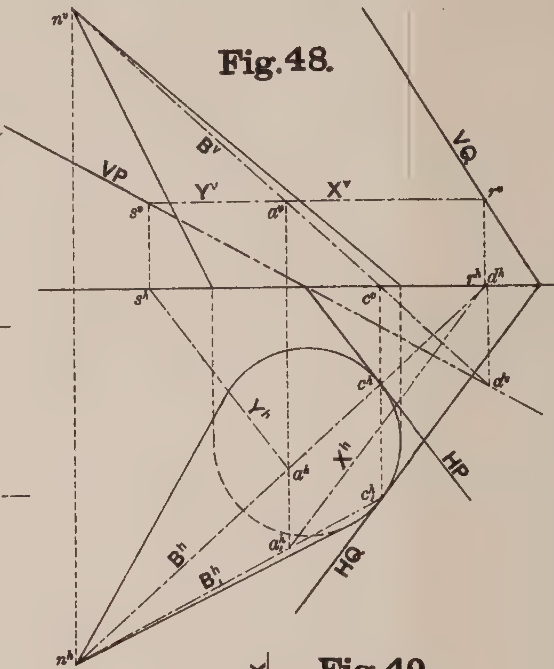

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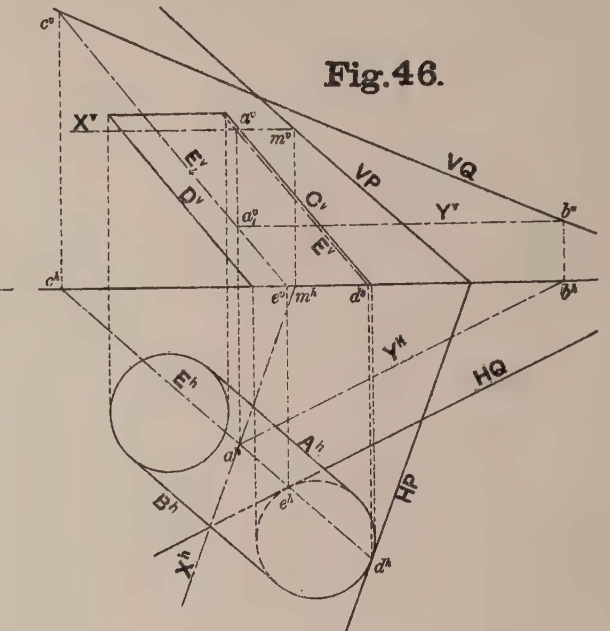

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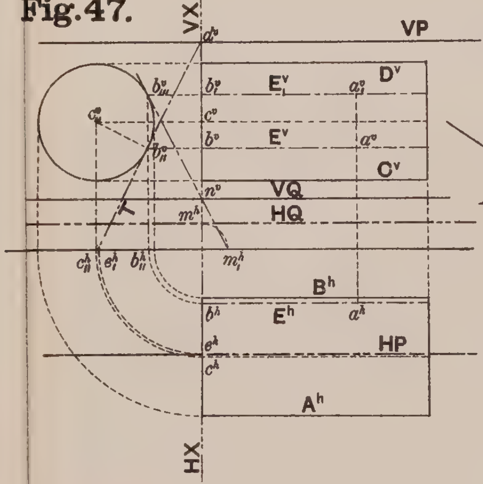

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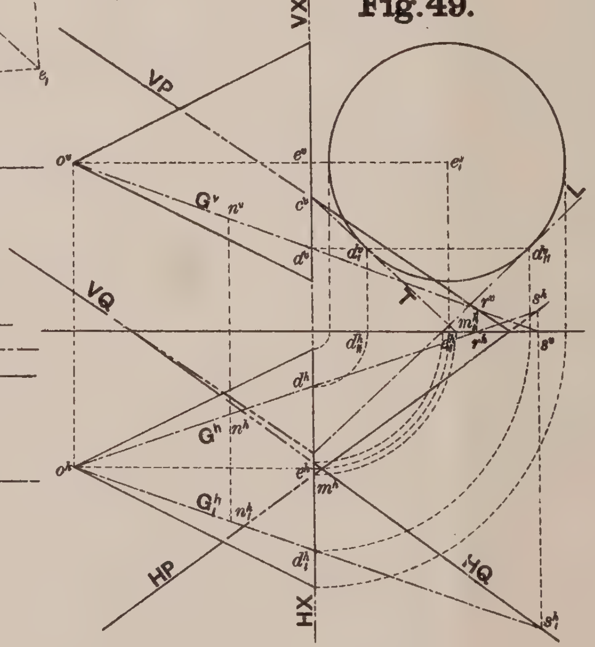

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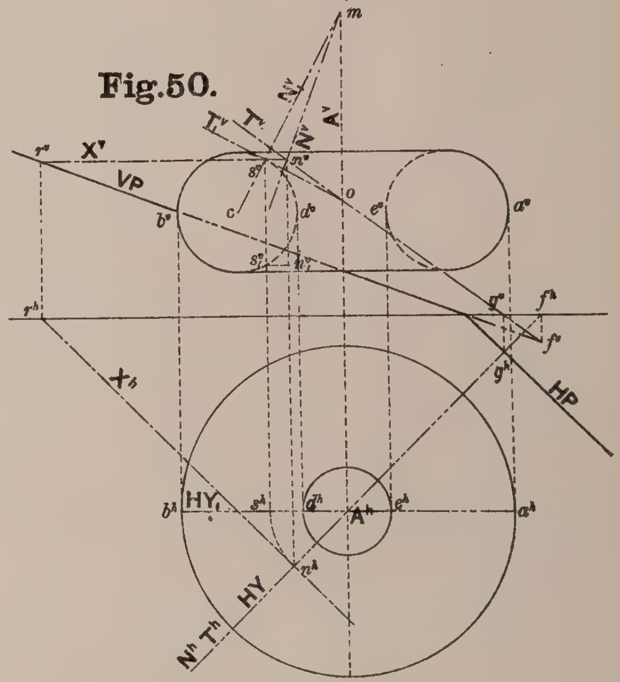

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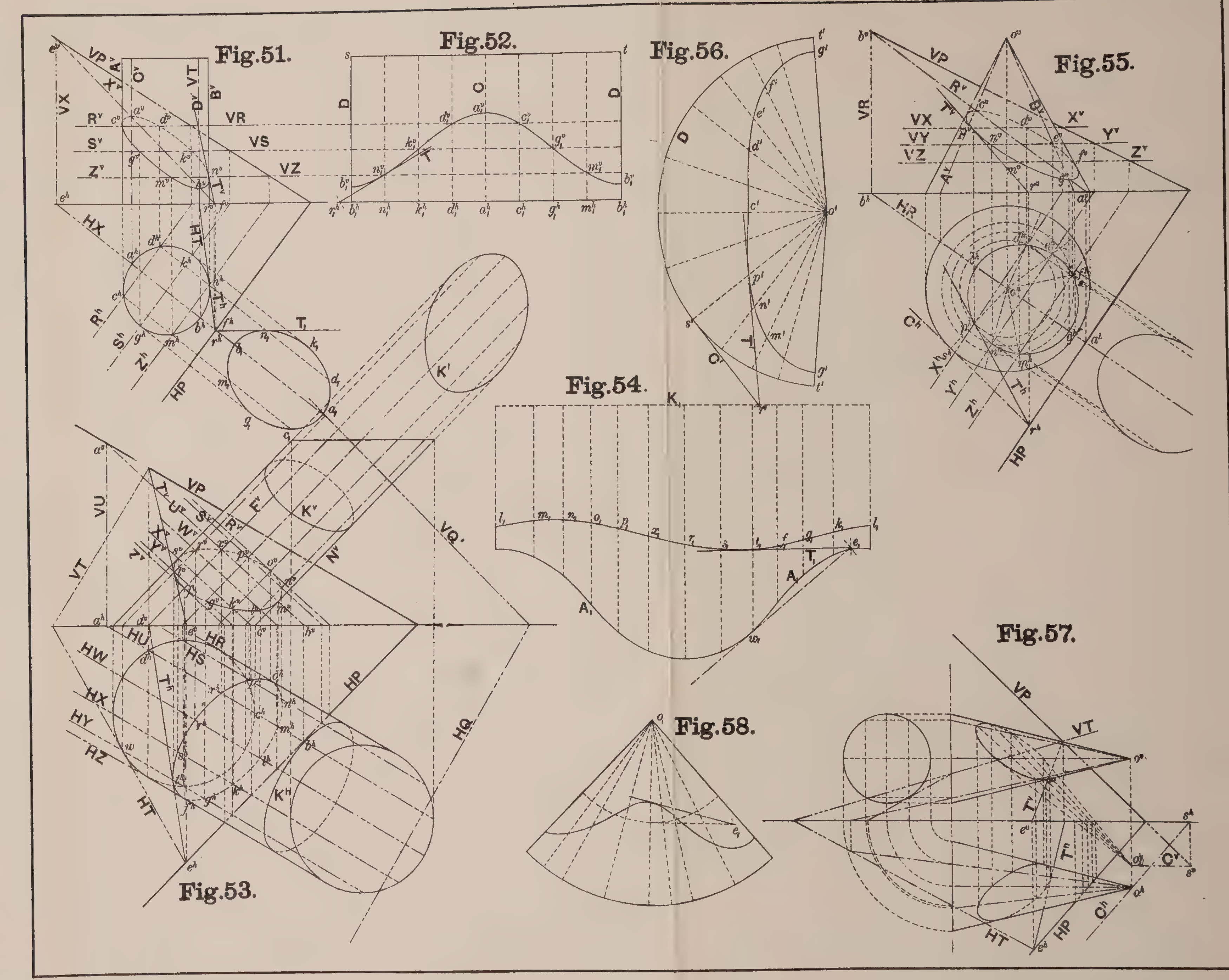
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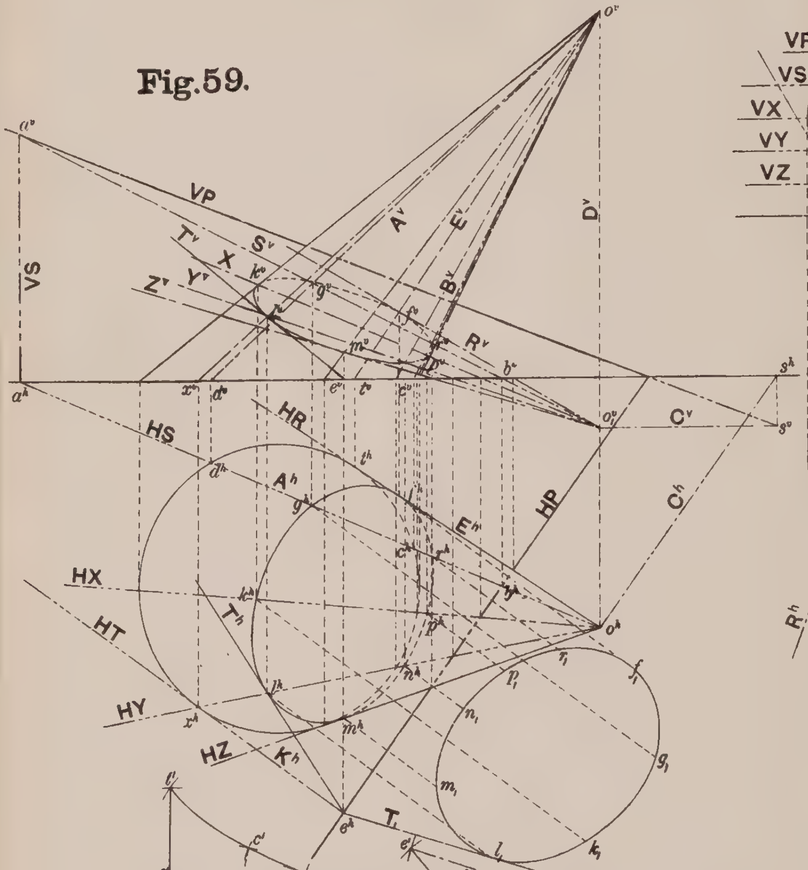
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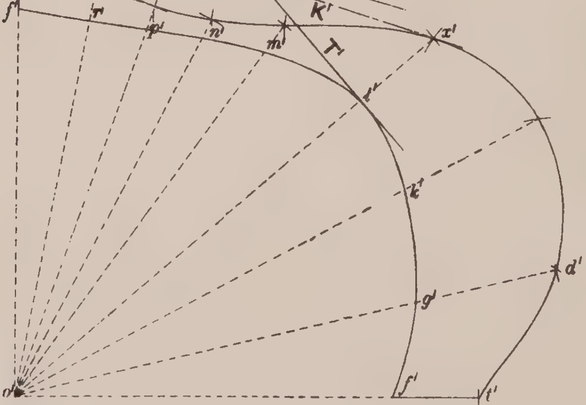
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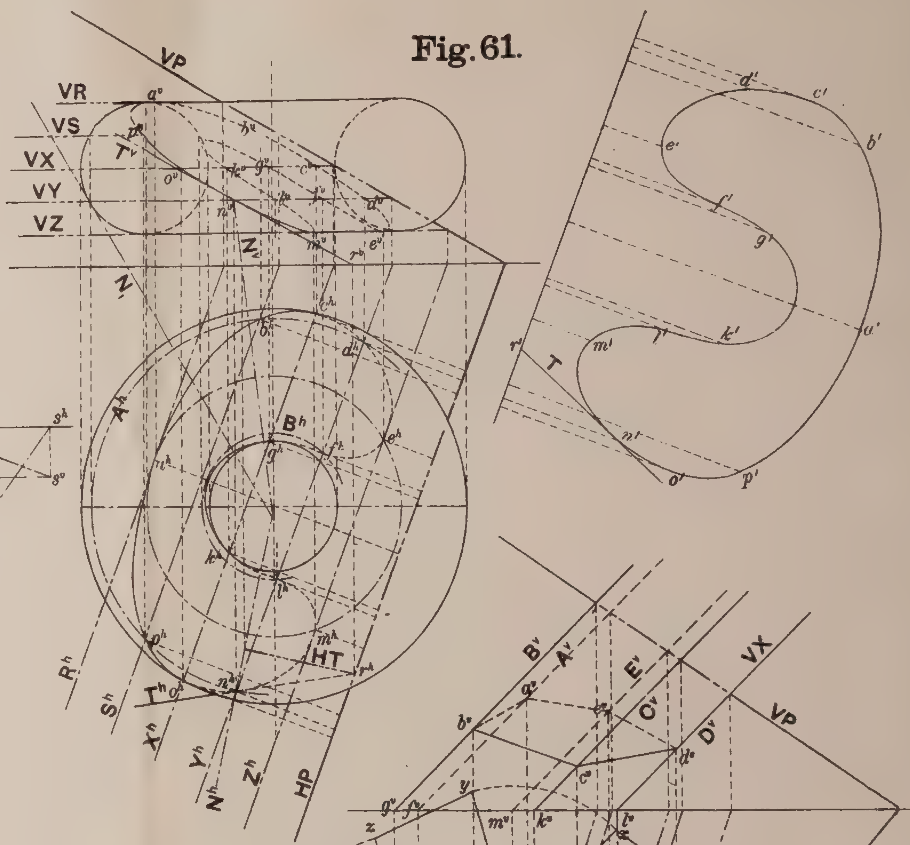
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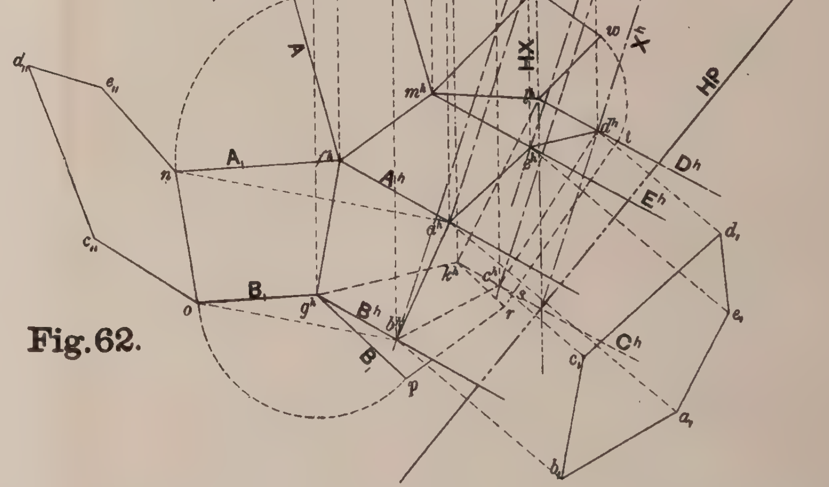
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Fig.63.

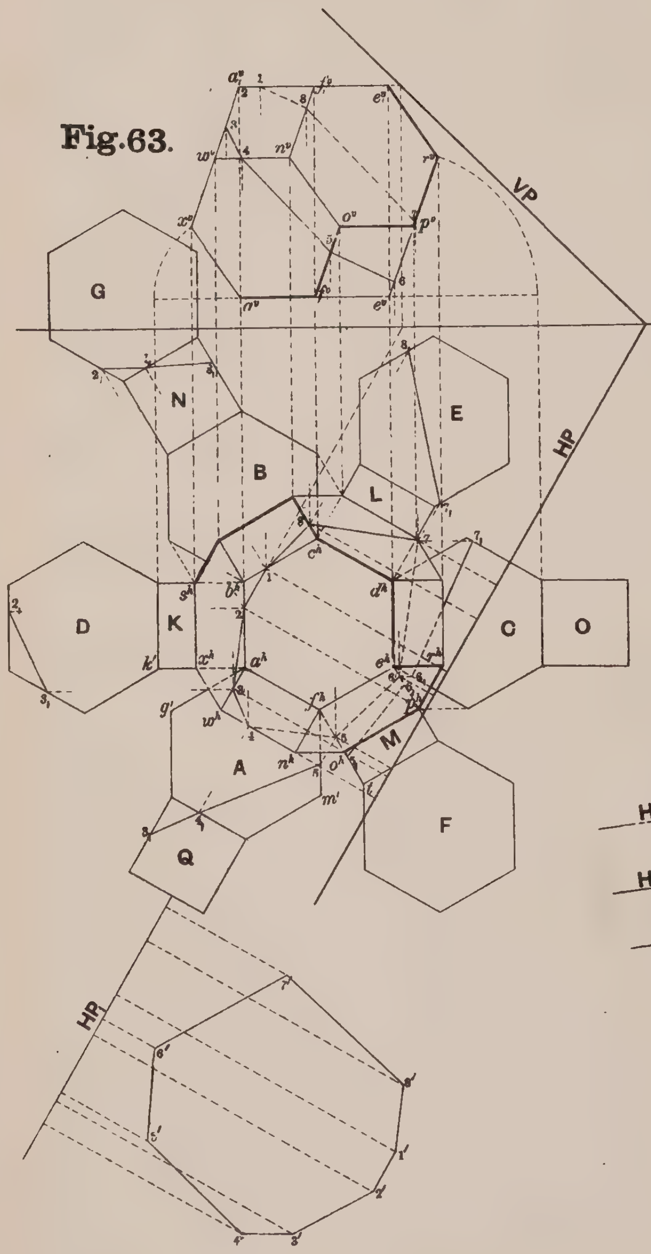
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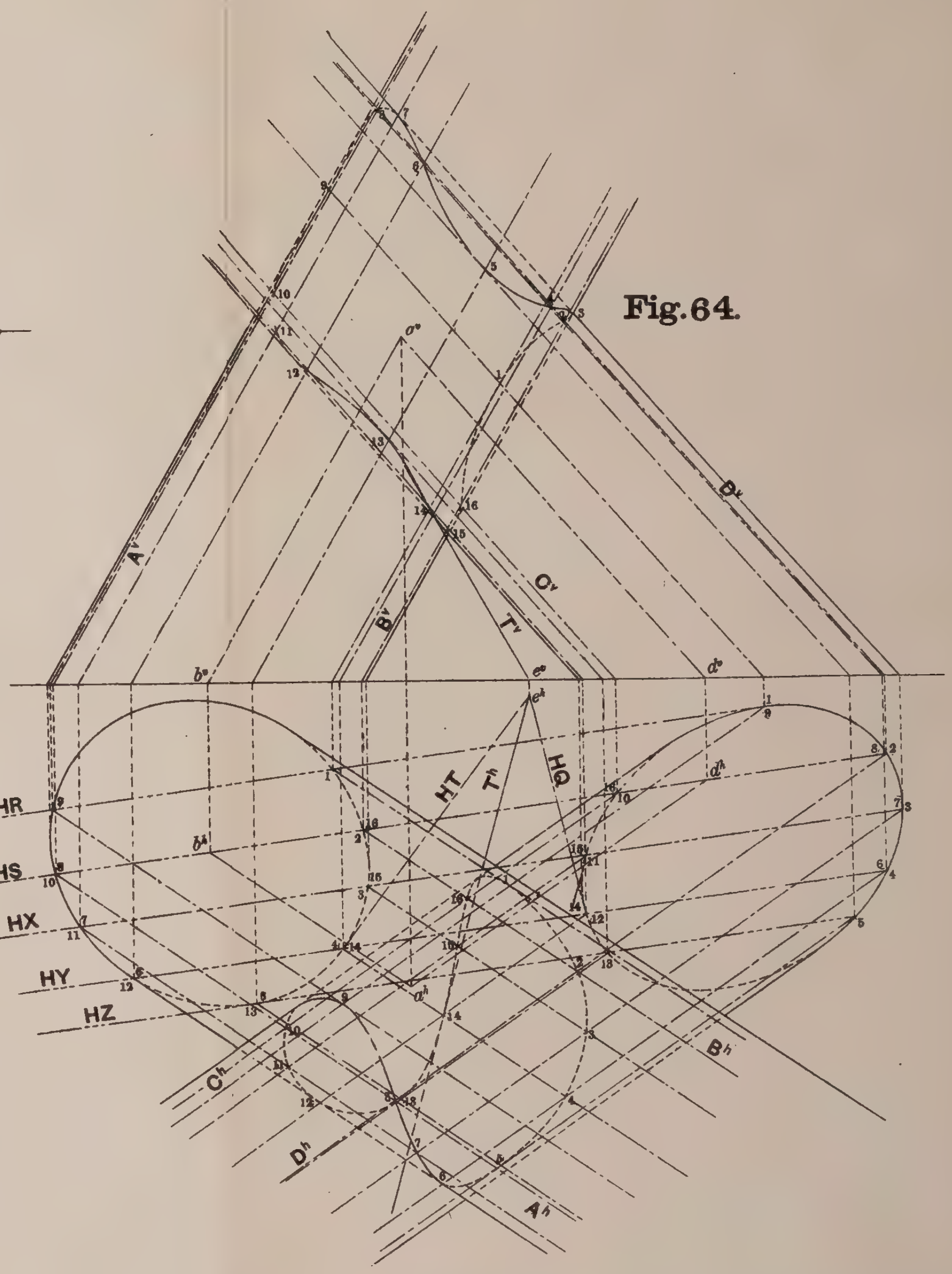
Fig.64.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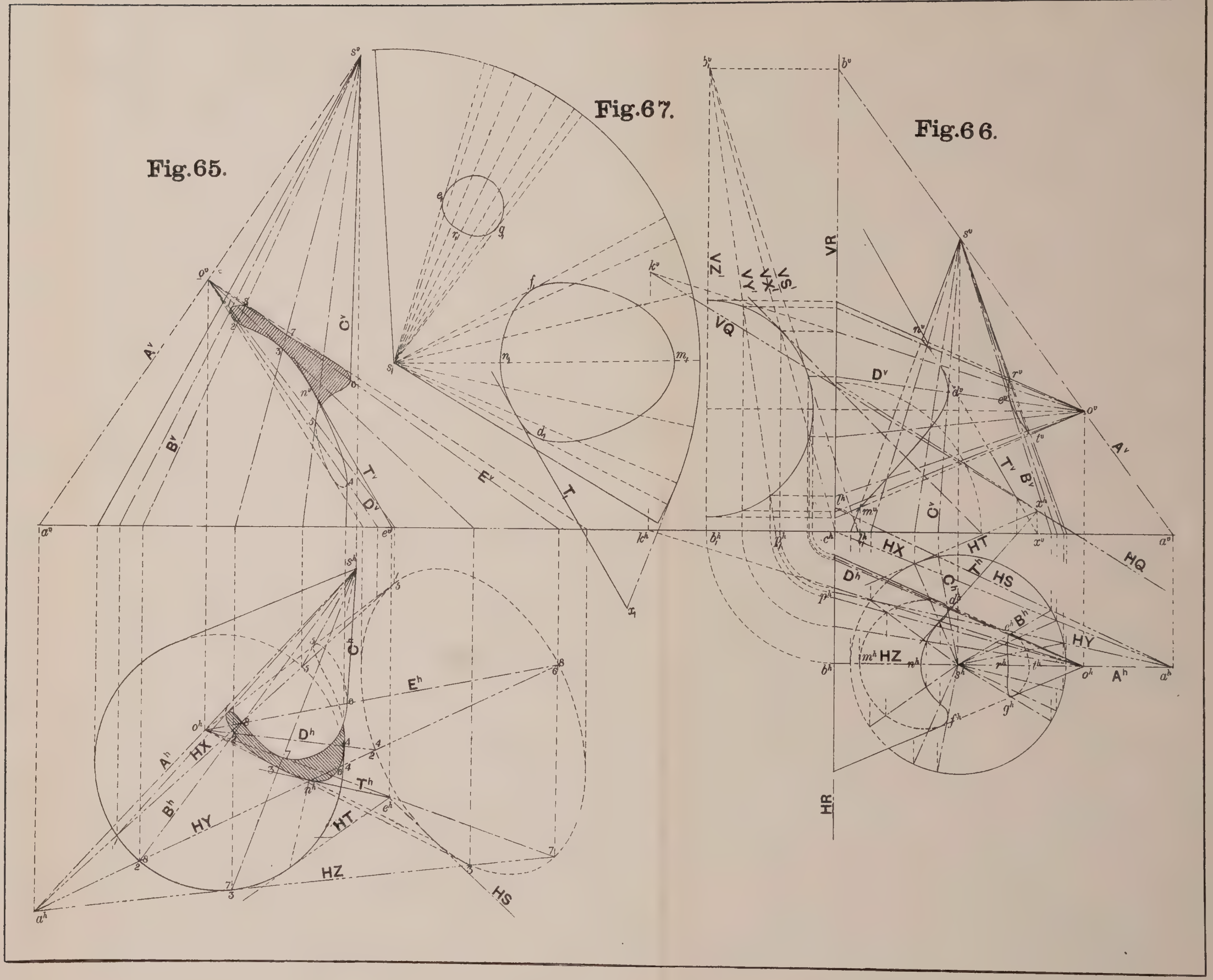

Fig.65.

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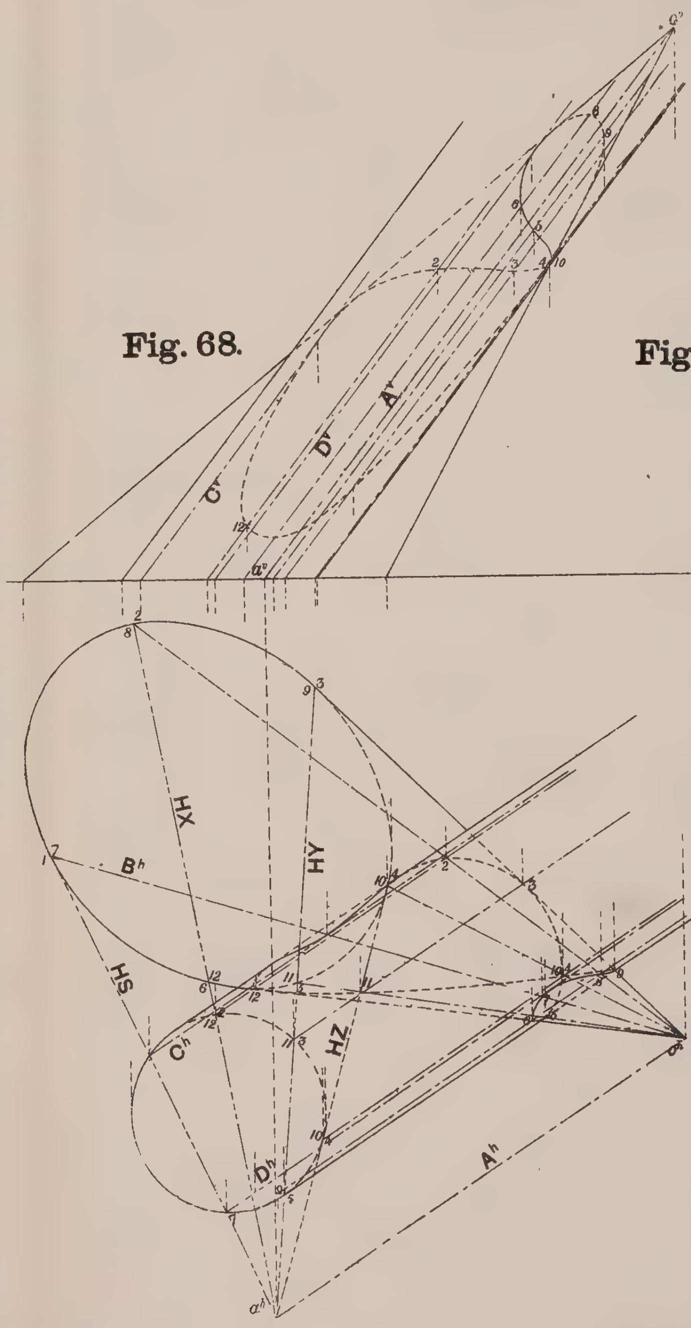
Fig.66.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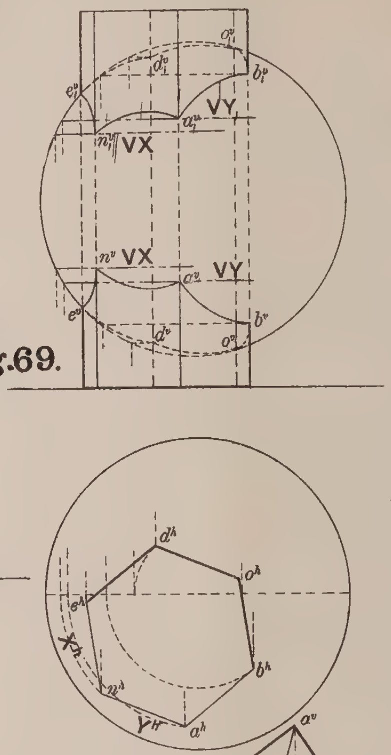
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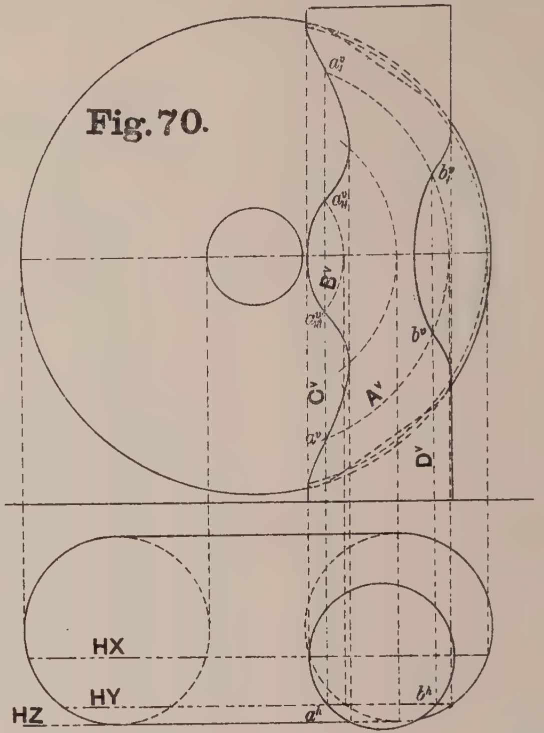
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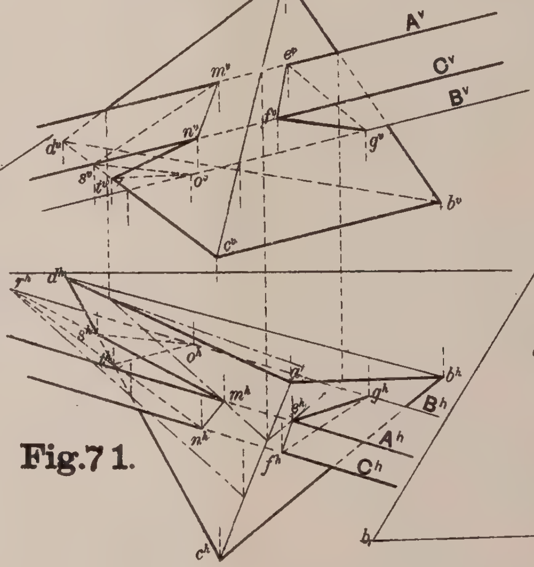
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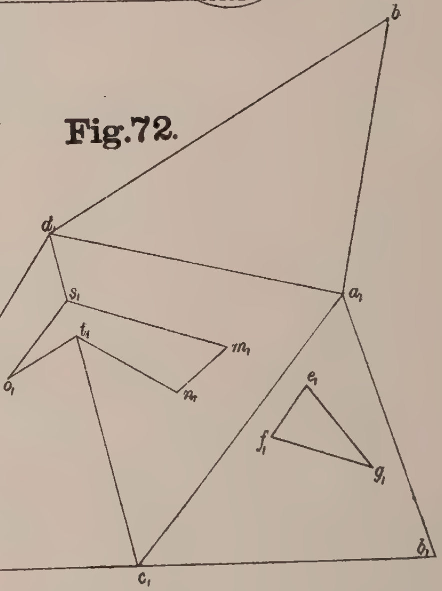
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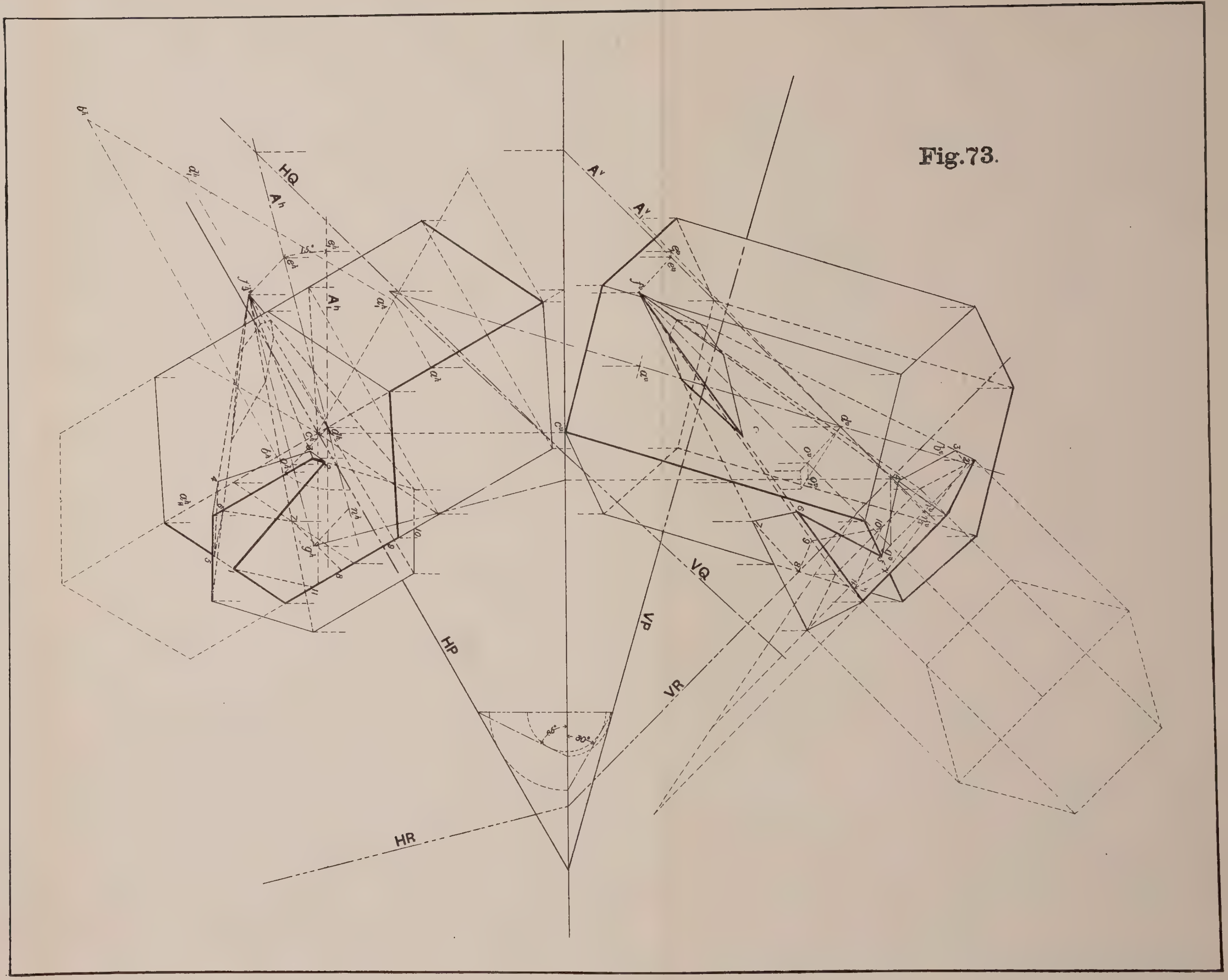

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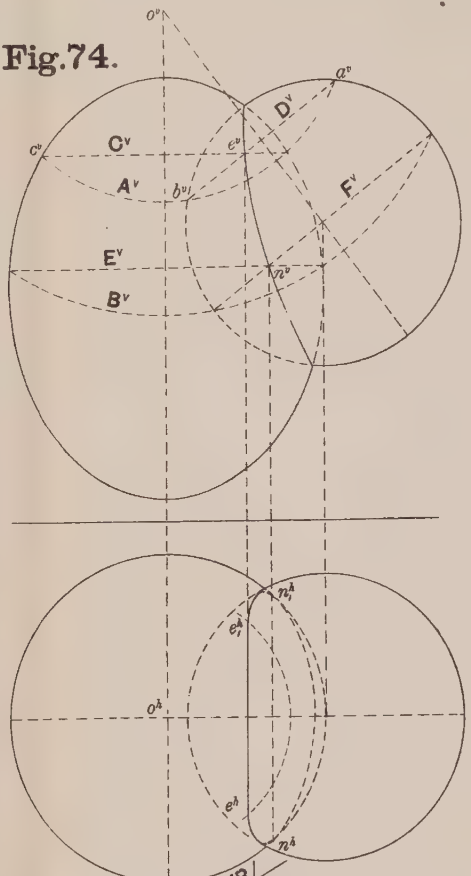

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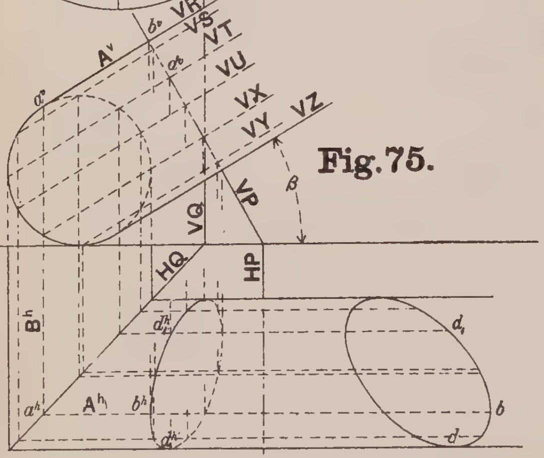

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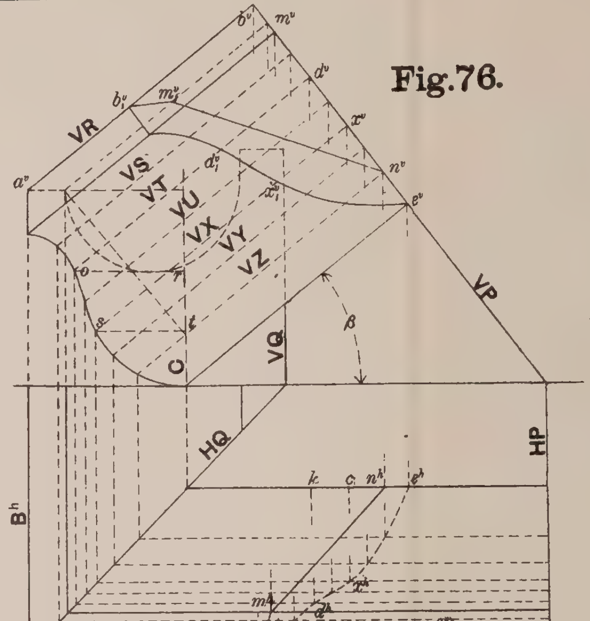

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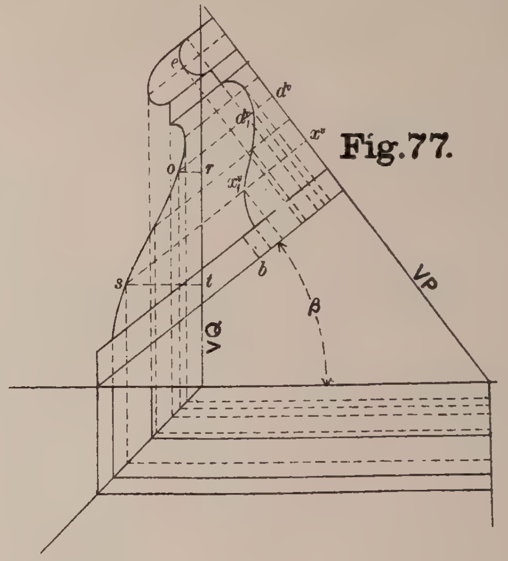

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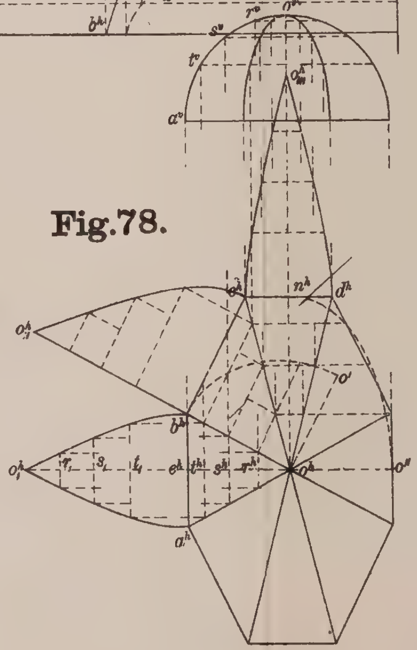

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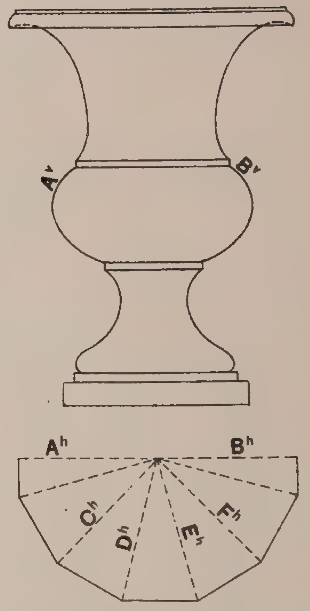

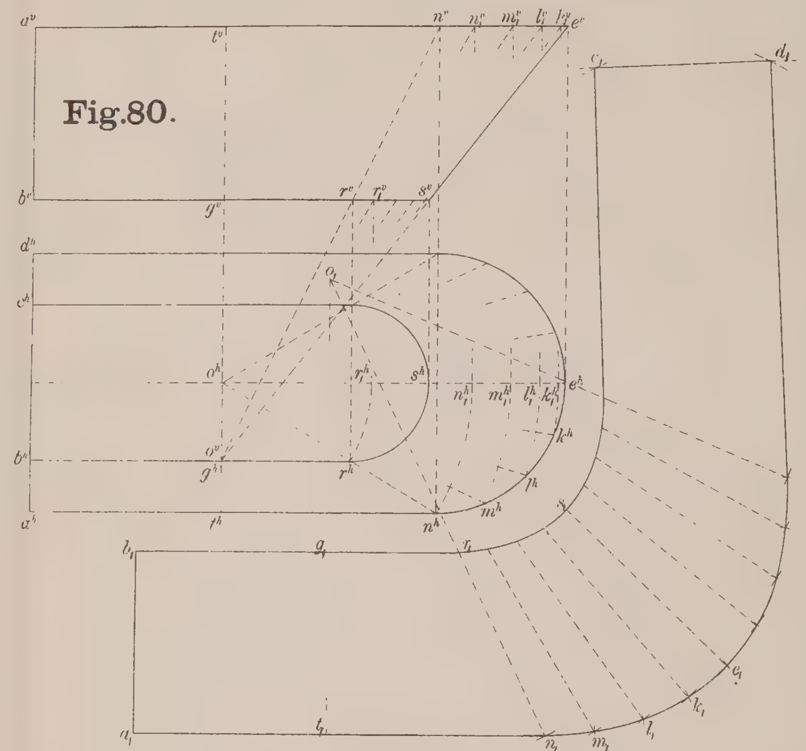

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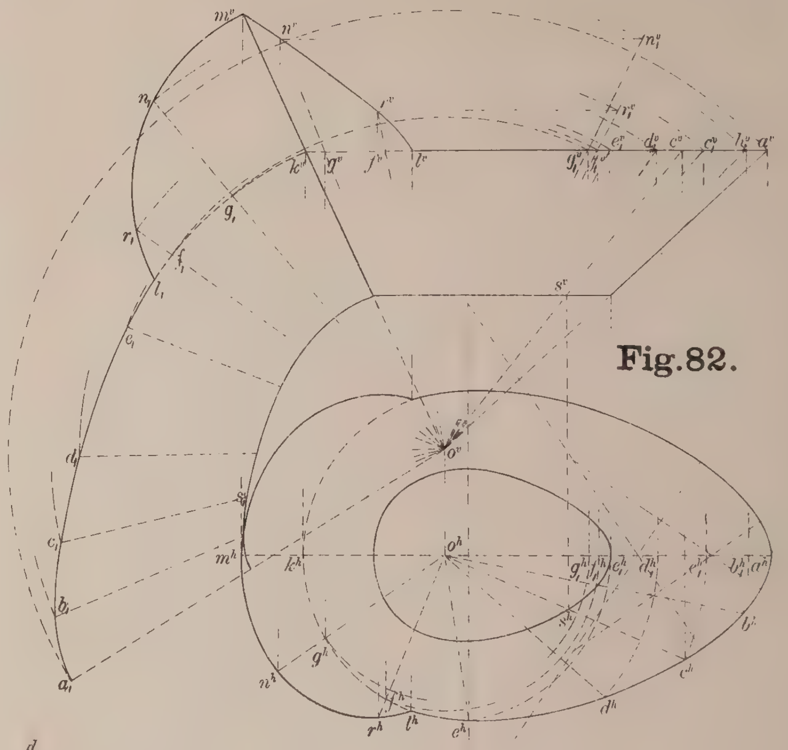

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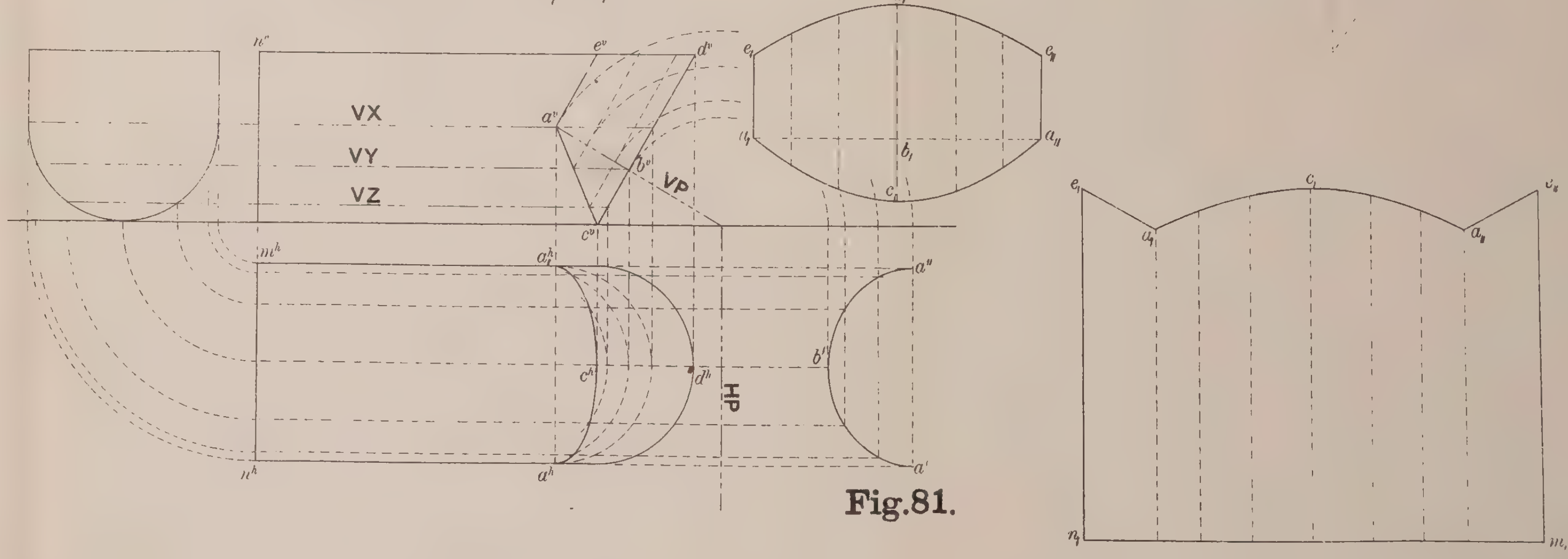

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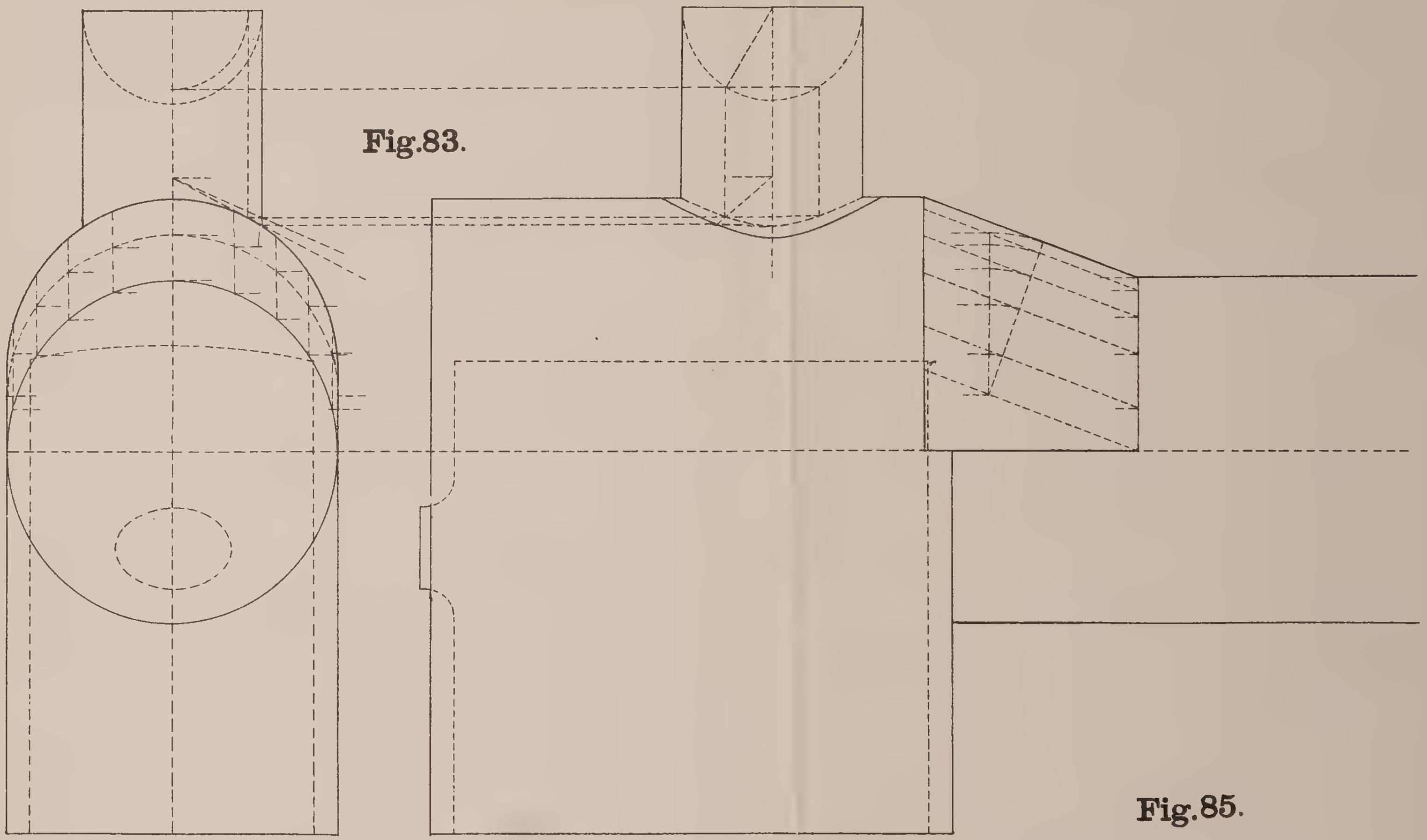

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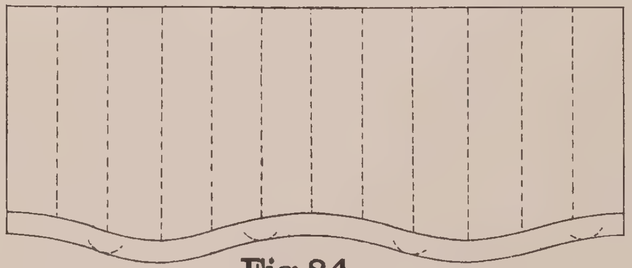
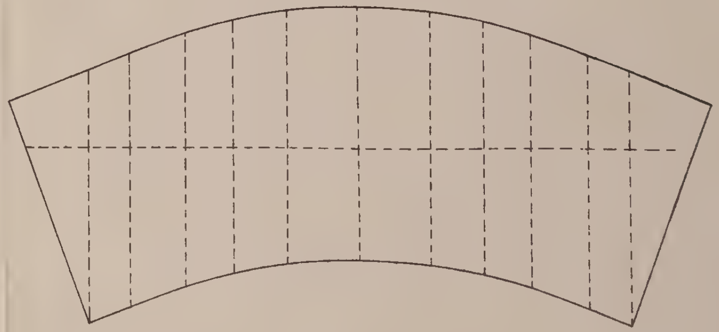


Fig.84.

Fig.85.



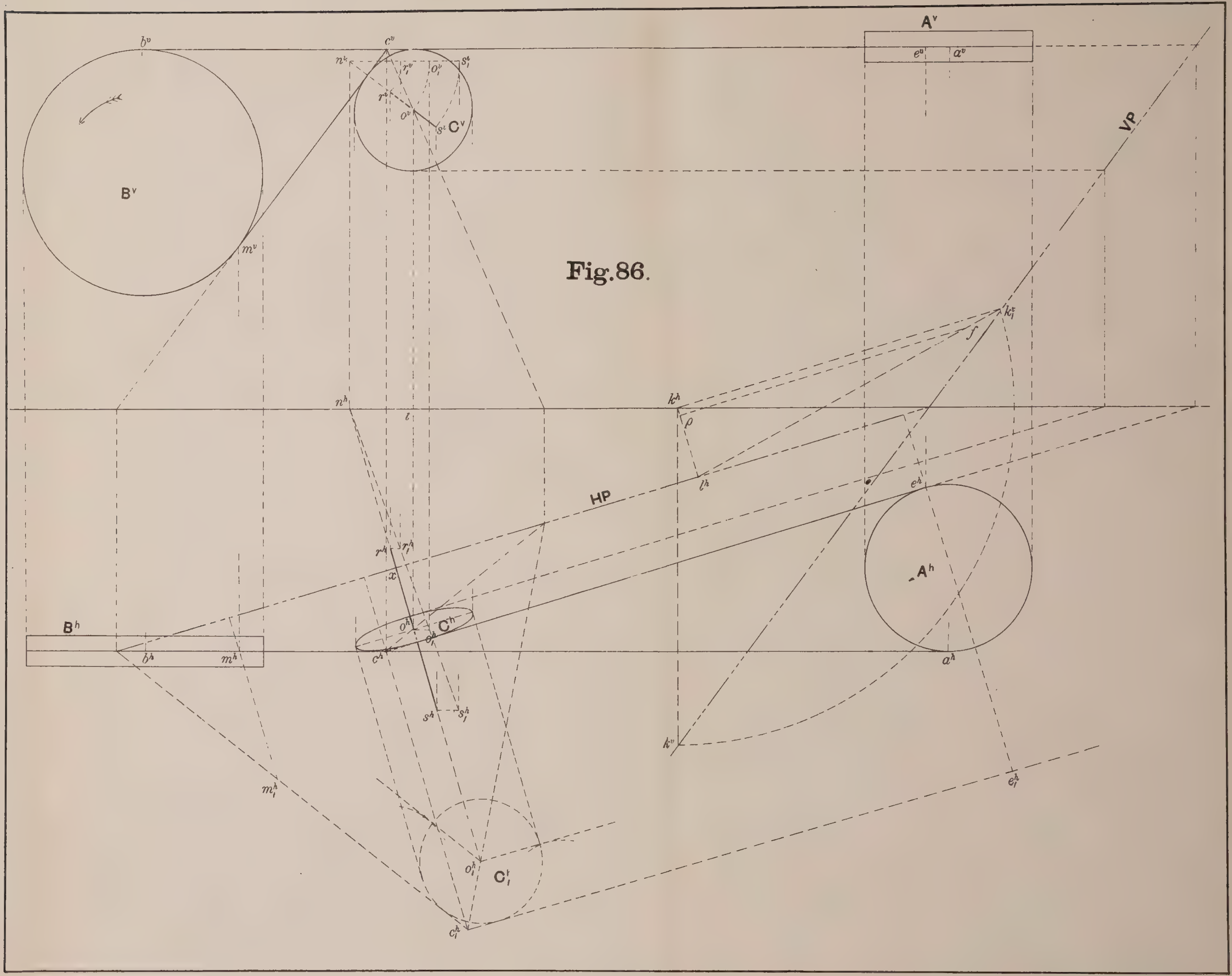

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Fig.86.

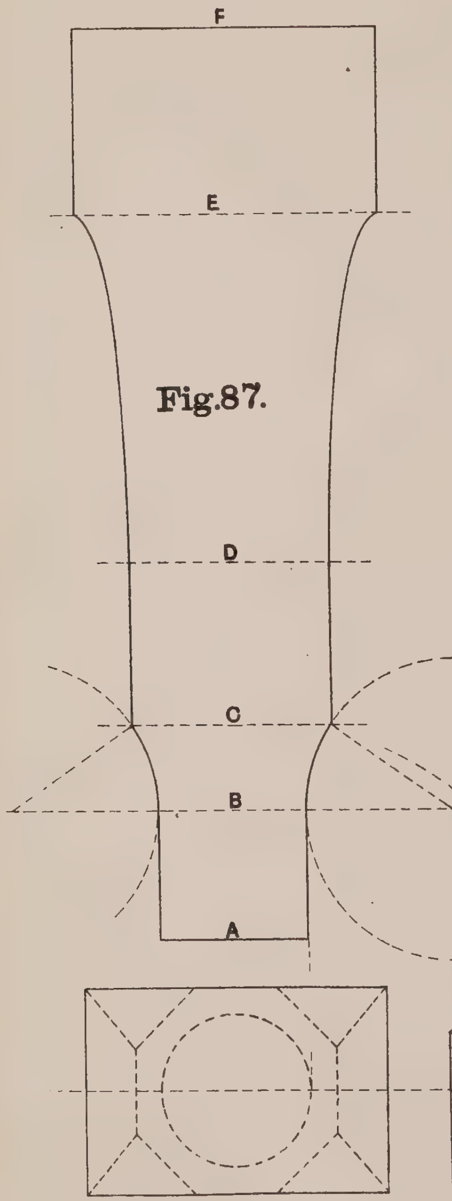

Fig.87.

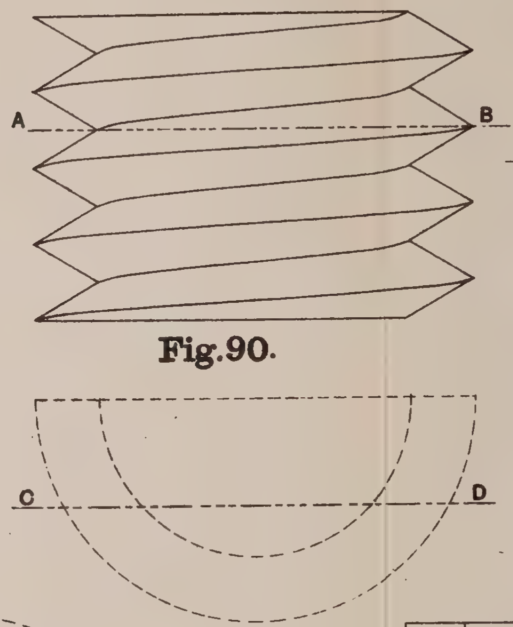

Fig.90.

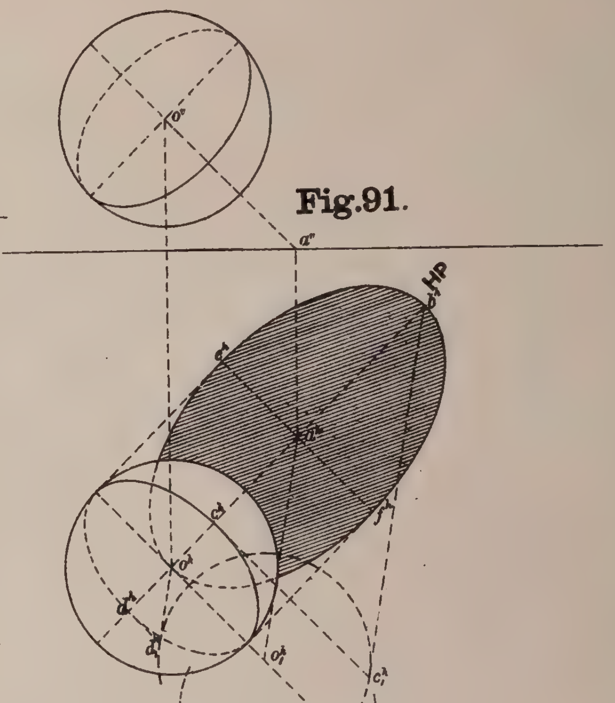

Fig.91.

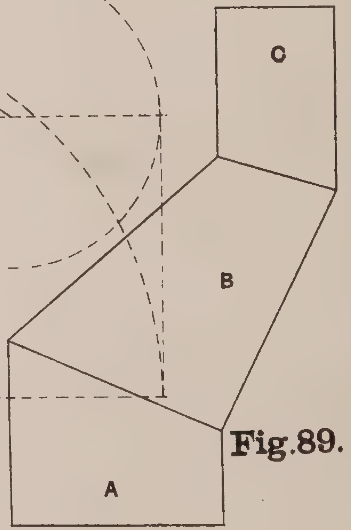

Fig.89.

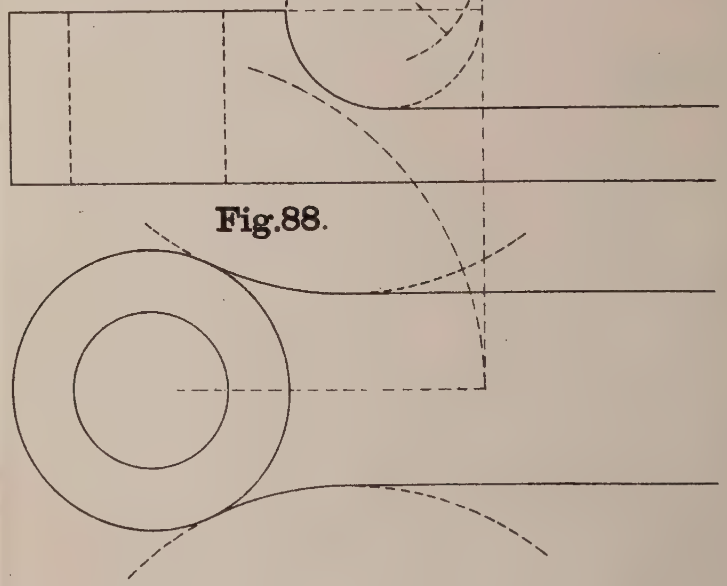

Fig.88.

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